Navier-Stokes Equations with Random Forcing (Stochastic Processes and Statistical Phenomena behind PDEs)

AUTHOR(S):

Yoshida, Nobuo

CITATION:

Yoshida, Nobuo. Navier-Stokes Equations with Random Forcing (Stochastic Processes and Statistical Phenomena behind PDEs). 数理解析研究所講究録 2013, 1823: 1-35

ISSUE DATE:

2013-01

URL:

http://hdl.handle.net/2433/194717

数理解析研究所講究録
第 1823 巻 2013 年 1-35

Navier-Stokes Equations with Random Forcing

Nobuo Yoshida[1]

Contents

0 Introduction

We would like to analyze the turbulence of a viscous fluid in $\mathbb{R}^d$ (physically, $d = 3$). Let

$$u = (u_i(t,x))_{i=1}^d \in \mathbb{R}^d \quad (0.1)$$

$$\Pi = \Pi(t,x) \in \mathbb{R} \quad (0.2)$$

[1]Division of Mathematics Graduate School of Science Kyoto University, Kyoto 606-8502, Japan. email: nobuo@math.kyoto-u.ac.jp URL: http://www.math.kyoto-u.ac.jp/~nobuo/

be the velocity and the pressure of the fluid at time $t \geq 0$ at the position $x \in \mathbb{R}^d$. For fluids like air and water, it is accepted in hydrodynamics that they satisfy the *Navier-Stokes equation*:

$$\mathrm{div} u = 0, \tag{0.3}$$

$$\partial_t u + (u \cdot \nabla)u = -\nabla \Pi + \nu \Delta u + F, \tag{0.4}$$

where $u \cdot \nabla = \sum_{j=1}^d u_j \partial_j$, $\nu > 0$ is a constant, called *kinematic viscosity*, and $F = F_t(x)$, $(t, x) \in [0, \infty) \times \mathbb{T}^d$ is a given external force. Physical interpretation of (0.3) is the mass conservation, while (0.4) is the motion equation.

On the other hand, since the turbulence is a random phenomenon, we need to bring a certain random factor into the model. To do so, we consider a *colored noise* , which is "time derivative" of a certain function space valued Brownian motion $W = W_t(x)$ and take $F_t(x) = \partial_t W_t(x)$ in (0.4). This may look too much of an idealization of the real turbulence. However, this way of modeling is common in literatures [Fl08] and references therein.

Based mainly on [Fl08], we explain the construction of the weak solution to (0.3)–(0.4) globally in time in the case $F_t(x) = \partial_t W_t(x)$.

1 Physical derivation of the Navier-Stokes equation

We review the heuristic argument to "derive" (0.3)–(0.4) from the physical assumptions. Let $e_1, .., e_d$ be the canonical basis of $\mathbb{R}^d$:

$$e_1 = (1, 0, ..., 0), \ e_2 = (0, 1, 0, ..., 0), \ ..., \ e_d = (0, ..., 0, 1). \tag{1.1}$$

Also, it is convenient to introduce the following small box and plaquettes:

$$\Box = \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]^d, \quad \Box_i = \{x \in \Box \ ; \ x_i = 0\}, \ \ i = 1, .., d, \tag{1.2}$$

where the side-length $\delta > 0$ of the box $\Box$ and the plaquette $\Box_i$ is supposed to be very small, eventually tending to zero. Let

$$u = (u_i(t, x))_{i=1}^d, \quad \rho = \rho(t, x) \geq 0 \tag{1.3}$$

be the velocity and the density of the fluid at time-space (t, x).

1.1 The mass conservation

We first derive (0.3) for a constant density fluid $\rho \equiv$ const. To do so, however, we *do not* assume that $\rho \equiv$ const. for a moment and consider the mass $m(x + \Box)$ of the fluid on the cube $x + \Box$ centered at x (cf. (1.2)):

$$m(x + \Box) = \int_{x+\Box} \rho \cong \rho(x)\delta^d \tag{1.4}$$

Here and often in what follows, we omit the time t in the notation. The time derivative of the mass is given as follows:

$$\partial_t m(x + \Box) = \sum_{j=1}^d m_j(x), \tag{1.5}$$

where

$$m_j(x) = \underbrace{(\rho u_j)\left(x - \frac{\delta}{2}e_j\right)\delta^{d-1}}_{\substack{\text{inward flux of the mass} \\ \text{through the face } (x - \frac{\delta}{2}e_j) + \Box_j}} - \underbrace{(\rho u_j)\left(x + \frac{\delta}{2}e_j\right)\delta^{d-1}}_{\substack{\text{outerward flux of the mass} \\ \text{through the face } (x + \frac{\delta}{2}e_j) + \Box_j}}$$

By Taylor expanding $(\rho u_j)\left(x \mp \frac{\delta}{2}e_j\right)$ above, we see that

$$\begin{aligned} m_j(x) &= \left((\rho u_j)(x) - \partial_j(\rho u_j)(x)\frac{\delta}{2} + O(\delta^2)\right)\delta^{d-1} \\ &- \left((\rho u_j)(x) + \partial_j(\rho u_j)(x)\frac{\delta}{2} + O(\delta^2)\right)\delta^{d-1} \\ &= -\partial_j(\rho u_j)(x)\delta^d + O(\delta^{d+1}). \end{aligned}$$

By this and (1.5), we get:

$$\frac{1}{\delta^d}\partial_t m(x + \Box) = -\sum_{j=1}^{d} \partial_j(\rho u_j)(x) + O(\delta) \tag{1.6}$$

Note that

$$\rho(x) = \lim_{\delta \searrow 0} \frac{1}{\delta^d} m(x + \Box).$$

If we believe that the above limit commutes with ∂_t, we see from (1.6) that

$$\partial_t \rho + \sum_{j=1}^{d} \partial_j(\rho u_j)(x) = 0. \tag{1.7}$$

In particular, for a constant density flow: $\rho \equiv$ const, (1.7) is reduced to (0.3). Note also that the interchange of the order of $\lim_{\delta \searrow 0}$ and ∂_t assumed above is perfectly correct in this case.

1.2 Force exerted on fluids: the stress tensor

The notion of *stress* can be thought of as actions, like pushing, pulling and rubbing a door. Then, the action has obviously different effects depending on the side of the door which the action is made on. Therefore, we distinguish the side of the plaqutte $\Box_i$: let

$$\begin{aligned} \Box_i^+ &= \text{“the } x_i > 0\text{-side” of } \Box_i = \{x \in \Box \; ; \; x_i = 0\} \\ \Box_i^- &= \text{the “opposite side” of } \Box_i. \end{aligned}$$

Imagine that the plaquette $\Box_i$ is put in a stream with the velocity u. Then forces are exerted on plane $\Box_i$, e.g., pulling, pushing, or rubbing. With this in mind, we introduce:

$$\begin{aligned} \tau_i^{\Box}(x) = (\tau_{ij}^{\Box}(x))_{j=1}^d &= \text{the force exerted on } x + \Box_i^+ \text{ by the stream} && (1.8) \\ &= -\text{the force exerted on } x + \Box_i^- \text{ by the stream,} && (1.9) \end{aligned}$$

where the second equality is, of course, the principle of action-reaction. We then define the *stress tensor* $\tau(x) = (\tau_{ij}(x))_{i,j=1}^d$ by:

$$\tau_{ij}(x) = \lim_{\delta \searrow 0} \frac{1}{\delta^{d-1}} \tau_{ij}^{\square}(x). \tag{1.10}$$

$\tau_{ij}(x)$ is the j-th component of the force exerted on x by the stream from the side x_i+. We will assume that

- τ is of the form:

$$\tau(x) = -\Pi(x)I + \tau^{\mathrm{F}}(x), \tag{1.11}$$

  where $\Pi(x) = \Pi(t,x)$ is the the pressure (a real function), I is the identity matrix, and $\tau^{\mathrm{F}}(x)$ is the *the friction term* of $\tau(x)$.

- τ is symmetric, i.e., $\tau_{ij} = \tau_{ji}$, or equivalently, $\tau_{ij}^{\mathrm{F}} = \tau_{ji}^{\mathrm{F}}$.

The symmetry assumption above is based on the conservation of the angular momentum. A typical example of the friction term is provided by the following *Stokes law*:

$$\tau_{ij}^{\mathrm{F}} = \mu\left(\partial_i u_j + \partial_j u_i\right), \tag{1.12}$$

where the constant $\mu > 0$ is the coefficient of friction, and the tensor $\left(\frac{\partial_i u_j + \partial_j u_i}{2}\right)$ is called the *symmetrized velocity gradient tensor*.

Let

$$f^{\square}(x) = \left(f_j^{\square}(x)\right)_{j=1}^d \text{ the force exerted on the outer boundary of } x + \square \text{ by the stream.}$$

Here, the outer boundary is the union of

$$(x + \frac{\delta}{2}e_i) + \square_i^+, \ (x - \frac{\delta}{2}e_i) + \square_i^- \ \ i = 1,..d.$$

Then, it turn out to be reasonable to define the force exerted to a point x by the stream by:

$$f(x) = (f_j(x))_{j=1}^d, \ \text{ where } \ f_j(x) = \lim_{\delta \searrow 0} \frac{1}{\delta^d} f_j^{\square}(x). \tag{1.13}$$

It may appear at first sight that "$2d\delta^{d-1}$" is more appropriate in place of δ^d above. However, we will see later on that δ^d is indeed the right normalization. We will prove that

$$f_j = \sum_{i=1}^d \partial_i \tau_{ij}. \tag{1.14}$$

Before we prove (1.14), we make some remarks. By (1.11), (1.14) reads:

$$f = -\nabla \Pi + \left(\sum_{i=1}^d \partial_i \tau_{ij}^{\mathrm{F}}\right)_{j=1}^d. \tag{1.15}$$

Moreover, if we suppose that the fluid is of constant density and the Stokes law (1.12) holds, then, since $\text{div}u = 0$,

$$\sum_{i=1}^{d} \partial_i \tau_{ij}^{\mathrm{F}} = \mu \sum_{i=1}^{d} (\partial_i \partial_i u_j + \partial_i \partial_j u_i) = \mu \Delta u_j.$$

Thus, (1.15) becomes:

$$f(x) = -\nabla \Pi + \mu \Delta u. \tag{1.16}$$

We turn to the proof of (1.14). We have , by (1.8)–(1.10) that

$$\begin{aligned} f_j^{\square}(x) &= \sum_{i=1}^{d} \underbrace{\tau_{ij}^{\square}\left(x + \frac{\delta}{2} e_i\right)}_{\substack{\text{the force exerted on} \\ (x + \frac{\delta}{2} e_i) + \square_i^+}} + \sum_{i=1}^{d} \underbrace{-\tau_{ij}^{\square}\left(x - \frac{\delta}{2} e_i\right)}_{\substack{\text{the force exerted on} \\ (x - \frac{\delta}{2} e_i) + \square_i^-}} \\ &\cong \sum_{i=1}^{d} \left(\tau_{ij}\left(x + \frac{\delta}{2} e_i\right) - \tau_{ij}\left(x - \frac{\delta}{2} e_i\right)\right) \delta^{d-1}. \end{aligned} \tag{1.17}$$

On the other hand, by Taylor expanding $\tau_{ij}\left(x \pm \frac{\delta}{2} e_i\right)$ above, we have that

$$\begin{aligned} &\tau_{ij}\left(x + \frac{\delta}{2} e_i\right) - \tau_{ij}\left(x - \frac{\delta}{2} e_i\right) \\ &= \left(\tau_{ij}(x) + \partial_i \tau_{ij}(x)\frac{\delta}{2} + O(\delta^2)\right) - \left(\tau_{ij}(x) - \partial_i \tau_{ij}(x)\frac{\delta}{2} + O(\delta^2)\right) \\ &= \partial_i \tau_{ij}(x)\delta + O(\delta^2). \end{aligned}$$

Plugging this into (1.17), we have

$$f_j^{\square}(x) \cong \partial_i \tau_{ij}(x)\delta^d + O(\delta^{d+1})$$

Thus, if we believe that the approximation $\cong$ is good enough, we have (1.14).

1.3 The motion equation

To derive the motion equation (0.4), we introduce the *stream line* $x(t) \in \mathbb{R}^d$, $t \geq 0$ define by:

$$x(t) = x(0) + \int_0^t u(s, x(s)) ds.$$

The curve $x(\cdot)$ is the integral curve of the velocity u, hence, roughly speaking, it is a position of a particle moving on the stream. The classical Newton's motion equation is:

$$\text{mass} \times \text{acceleration} = \text{force},$$

which, in our case, takes the following form:

$$\rho(x(t)) \frac{d}{dt} u(t, x(t)) = f(x(t)), \tag{1.18}$$

where the force f is given by (1.15). We have by the chain rule that

$$\begin{aligned}\frac{d}{dt}u(t,x(t)) &= \partial_t u(t,x(t)) + \sum_{j=1}^d \partial_j u(t,x(t)) \underbrace{\frac{dx_j(t)}{dt}}_{u_j(t,x(t))} \\ &= (\partial_t u + (u\cdot\nabla)u)\,(t,x(t)).\end{aligned}$$

By the above identity, together with (1.15) and (1.18), we get

$$\rho(\partial_t u + (u\cdot\nabla)u) = -\nabla\Pi + \left(\sum_{i=1}^d \partial_i \tau_{ij}^{\mathrm{F}}\right)_{j=1}^d. \tag{1.19}$$

If we suppose that the fluid is of constant density and the Stokes law (1.12) holds, then, by (1.16), we have that

$$\partial_t u + (u\cdot\nabla)u = -\frac{1}{\rho}\nabla\Pi + \frac{\mu}{\rho}\Delta u, \tag{1.20}$$

where the constant $\nu \stackrel{\text{def}}{=} \frac{\mu}{\rho}$ is the kinematic viscosity.

2 The mathematical framework in the case of non-random forcing term

From here on, we assume that the container of the fluid is the d-dimensional torus:

$$\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0,1]^d.$$

This is a part of idealization. The unknown functions of the Navier-Stokes equation **(NS)** are

- ▶ *velocity of fluid* $u = u_t(x) \in \mathbb{R}^d$, $(t,x) \in [0,\infty)\times\mathbb{T}^d$ with suitable regularity, say C^2 in (t,x).

- ▶ *pressure* $\Pi = \Pi_t(x) \in \mathbb{R}$, $(t,x) \in [0,\infty)\times\mathbb{T}^d$ with suitable regularity, say C^1 in (t,x).

Given an initial velocity $u_0 : \mathbb{T}^d \to \mathbb{R}^d$,

$$\mathrm{div} u = 0, \tag{2.1}$$
$$\partial_t u + (u\cdot\nabla)u = -\nabla\Pi + \nu\Delta u + F, \tag{2.2}$$

where $\nu > 0$ is a constant, called *kinematic viscosity* and $F = F_t(x)$, $(t,x) \in [0,\infty)\times\mathbb{T}^d$ is a given external force. Physical interpretation of (2.1) and (2.2) were explained in section 1.

2.1 A weak formulation

Let $\mathcal{V}$ be the set of $\mathbb{R}^d$-valued divergence free, mean-zero trigonometric polynomials, i.e., the set of $v : \mathbb{T}^d \to \mathbb{R}^d$ of the following form:

$$v(x) = \sum_{z\in\mathbb{Z}^d} \widehat{v}_z \psi_z(x), \quad x \in \mathbb{T}^d, \tag{2.3}$$

where $\psi_z(x) = \exp(2\pi \mathbf{i} z \cdot x)$ and the coefficients $\widehat{v}_z \in \mathbb{R}^d$ satisfy

$$\begin{aligned} \widehat{v}_z &= 0 \quad \text{for } z = 0 \text{ and except for finitely many } z \neq 0, &(2.4)\\ \overline{\widehat{v}_z} &= \widehat{v}_{-z} \quad \text{for all } z, &(2.5)\\ z \cdot \widehat{v}_z &= 0 \quad \text{for all } z. &(2.6)\end{aligned}$$

Note that (2.6) implies that:

$$\mathrm{div} v = 0 \quad \text{for all } v \in \mathcal{V}.$$

We equip the torus $\mathbb{T}^d$ with the Lebesgue measure and denote by $\|f\|_p$ the usual L_p-norm of $f \in L_p(\mathbb{T}^d)$. For $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$ we define

$$(1-\Delta)^{\alpha/2} v = \sum_{z \in \mathbb{Z}^d} (1 + 4\pi^2 |z|^2)^{\alpha/2} \widehat{v}_z \psi_z.$$

We then introduce:

$$V_{2,\alpha} = \text{the completion of } \mathcal{V} \text{ with respect to the norm } \|\ \cdot\ \|_{2,\alpha}\,, \quad \alpha \in \mathbb{R}, \tag{2.7}$$

where

$$\|v\|_{2,\alpha}^2 = \int_{\mathbb{T}^d} |(1-\Delta)^{\alpha/2} v|^2 = \sum_{z \in \mathbb{Z}^d} (1 + 4\pi^2 |z|^2)^{\alpha} |\widehat{v}_z|^2. \tag{2.8}$$

Here are some basic properties of the space $V_{2,\alpha}$:

- Any $v \in V_{2,\alpha}$ is identified with a summation of the form (2.3) with (2.4) replaced by the condition that the last summation in (2.8) converges.
- $V_{2,-\alpha}$ is identified with the set of continuous linear functional on $V_{2,\alpha}$.
- $$V_{2,\alpha+\beta} \hookrightarrow\!\!\!\hookrightarrow V_{2,\alpha}, \quad \text{for } \alpha \in \mathbb{R} \text{ and } \beta > 0. \tag{2.9}$$

cf. Definition 2.1.1 and Exercise 2.1.1 below.

Definition 2.1.1 Let E_0, E_1 be normed vector spaces.

► $E_0 \hookrightarrow E_1$ means that E_0 is continuously imbeded into E_1, i.e., $E_0 \subset E_1$ with the inclusion map being continuous.

► $E_0 \hookrightarrow\!\!\!\hookrightarrow E_1$ means that E_0 is compactly imbeded into E_1, i.e., $E_0 \subset E_1$ with the inclusion map being a compact operator.

Exercise 2.1.1 Recall that any $v \in V_{2,\alpha}$ is identified with a summation of the form (2.3) with (2.4) replaced by the condition that the last summation in (2.8) converges. Let $\alpha \in \mathbb{R}$, $\beta > 0$ and $v \in V_{2,\alpha+\beta}$. Prove that

$$\|v - I_n v\|_{2,\alpha} \leq (1 + 4\pi^2 n^2)^{-\beta/2} \|v\|_{2,\alpha+\beta}, \quad \text{where } I_n v = \sum_{|z| \leq n} \widehat{v}_z \psi_z.$$

Then, conclude (2.9) from this.

Exercise 2.1.2 Prove the following interpolation inequality:

$$\|v\|_{2,\theta\alpha+(1-\theta)\beta} \leq \|v\|_{2,\alpha}^{\theta} \|v\|_{2,\beta}^{1-\theta} \quad \text{for } \alpha, \beta \in \mathbb{R} \text{ and } \theta \in [0,1]. \tag{2.10}$$

For $v, w : \mathbb{T}^d \to \mathbb{R}^d$, with w supposed to be differentiable (for a moment), we define a vector field:

$$(v \cdot \nabla)w = \sum_{i=1}^{d} v_i \partial_i w, \tag{2.11}$$

which is bilinear in (v, w). Later on, we will generalize the definition of the above vector field (cf. (2.18)).

Lemma 2.1.2 *For $v \in \mathcal{V}$, $w, \varphi \in C^1(\mathbb{T}^d \to \mathbb{R}^d)$,*

$$\langle \varphi, (v \cdot \nabla)w \rangle = -\langle w, (v \cdot \nabla)\varphi \rangle, \tag{2.12}$$

In particular, $\langle w, (v \cdot \nabla)w \rangle = 0$.

Proof: Since $\mathrm{div} v = 0$, we have that

1) $$\sum_j \partial_j(\varphi_i v_j) = \sum_j (\partial_j \varphi_i) v_j + \varphi_i \underbrace{\sum_j \partial_j v_j}_{=0}.$$

Therefore,

$$\begin{aligned}\text{LHS (2.12)} = \sum_{i,j} \langle \varphi_i, v_j \partial_j w_i \rangle &= -\sum_{i,j} \langle \partial_j(\varphi_i v_j), w_i \rangle \\ &\overset{1)}{=} -\sum_{i,j} \langle (\partial_j \varphi_i) v_j, w_i \rangle = \text{RHS (2.12)}.\end{aligned}$$

$\square$

Suppose that u, Π, F in (NS) ((2.1)–(2.2)) have suitable regularity. Then, for a test function $\varphi \in \mathcal{V}$,

$*)$ $$\partial_t \langle \varphi, u \rangle = -\underbrace{\langle \varphi, (u \cdot \nabla)u \rangle}_{(1)} + \nu \underbrace{\langle \varphi, \Delta u \rangle}_{(2)} - \underbrace{\langle \varphi, \nabla \Pi \rangle}_{(3)} + \langle \varphi, F \rangle.$$

$$(1) \overset{(2.12)}{=} -\langle u, (u \cdot \nabla)\varphi \rangle, \quad (2) = \langle \Delta\varphi, u \rangle, \quad (3) = -\langle \mathrm{div}\varphi, \Pi \rangle = 0.$$

Thus, $*)$ becomes

$$\partial_t \langle \varphi, u \rangle = \langle u, (u \cdot \nabla)\varphi \rangle + \nu \langle \Delta\varphi, u \rangle + \langle \varphi, F \rangle.$$

By integration, we arrive at:

$$\langle \varphi, u_t \rangle = \langle \varphi, u_0 \rangle + \int_0^t (\langle u_s, (u_s \cdot \nabla)\varphi \rangle + \nu \langle \Delta\varphi, u_s \rangle + \langle \varphi, F_s \rangle)\, ds. \tag{2.13}$$

This is a standard weak formulation of (NS) ((2.1)–(2.2)).

2.2 Bounds on the non-linear term

Lemma 2.2.1 *Suppose $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with at least two of them being non-zero, and that $\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{d}{2}$. Then, there exists $C \in (0, \infty)$ such that:*

$$|\langle\, w, (v \cdot \nabla)\varphi \,\rangle| \leq C\|v\|_{2,\alpha_1}\|w\|_{2,\alpha_2}\|\varphi\|_{2,1+\alpha_3}, \tag{2.14}$$

for $v, w, \varphi \in C^\infty(\mathbb{T}^d \to \mathbb{R}^d)$.

Proof: Since the norm $\|\cdot\|_{2,\alpha}$ is increasing in α, it is enough to prove (2.16) with α_i replaced by $\widetilde{\alpha}_i = \frac{(d/2)\alpha_i}{\alpha_1+\alpha_2+\alpha_3}$. Therefore, we may assume without loss of generality that

$$(\alpha_1, \alpha_2, \alpha_3) \in [0, \tfrac{d}{2})^3 \;\text{ and }\; \alpha_1 + \alpha_2 + \alpha_3 = \tfrac{d}{2}.$$

Let $q_i \in [2, \infty)$, $i = 1, 2, 3$ be defined by $\frac{1}{q_i} = \frac{1}{2} - \frac{\alpha_i}{d} > 0$. Since

$$\sum_{i,j} |w_i v_j \partial_j \varphi_i| \leq |w||v||\nabla\varphi|,$$

we have

$$|\langle\, w, (v \cdot \nabla)\varphi \,\rangle| \overset{\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1}{\leq} \|v\|_{q_1}\|w\|_{q_2}\|\nabla\varphi\|_{q_3}.$$

We then use the following Sobolev imbedding theorem (e.g.[Ta96, p.4, (2.11)]):

$$V_{2,\alpha} \hookrightarrow L_q(\mathbb{T}^d \to \mathbb{R}^d), \;\text{ if } \tfrac{1}{q} \overset{\text{def}}{=} \tfrac{1}{2} - \tfrac{\alpha}{d} > 0. \tag{2.15}$$

□

We have the following variant of Lemma 2.2.1, which is applicable even when $\alpha_2 = \alpha_3 = 0$:

Lemma 2.2.2 *Let $\alpha_1, \alpha_2, \alpha_3 \geq 0$ be such that $\alpha_1 + \alpha_2 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{d}{2}$. Then, there exists $C \in (0, \infty)$ such that:*

$$|\langle\, w, (v \cdot \nabla)\varphi \,\rangle| \leq C\|\varphi\|_{2,1+\alpha_3}\sqrt{\|v\|_{2,\alpha_1}\|v\|_{2,\alpha_2}\|w\|_{2,\alpha_1}\|w\|_{2,\alpha_2}}, \tag{2.16}$$

for $v, w, \varphi \in C^\infty(\mathbb{T}^d \to \mathbb{R}^d)$.

Proof: Note that

1) $\qquad \|u\|_{2,\frac{\alpha_1+\alpha_2}{2}} \overset{(2.10)}{\leq} \sqrt{\|u\|_{2,\alpha_1}\|u\|_{2,\alpha_2}} \quad$ for $u \in V_{2,\alpha_1} \cap V_{2,\alpha_2}$.

On the other hand, by (2.14) with $(\frac{\alpha_1+\alpha_2}{2}, \frac{\alpha_1+\alpha_2}{2}, \alpha_3)$ in place of $(\alpha_1, \alpha_2, \alpha_3)$, we have

$$|\langle\, w, (v \cdot \nabla)\varphi \,\rangle| \overset{(2.14)}{\leq} C\|v\|_{2,\frac{\alpha_1+\alpha_2}{2}}\|w\|_{2,\frac{\alpha_1+\alpha_2}{2}}\|\varphi\|_{2,1+\alpha_3} \overset{1)}{\leq} \text{RHS (2.16)}.$$

□

Remark: (2.16) gives a generalization of [Te79, p.292, Lemma 3.4]

Let

$$\alpha_1, \alpha_2 \geq 0, \;\; \alpha_1 + \alpha_2 > 0, \;\text{ and }\; \alpha_3 \overset{\text{def.}}{=} \left(\tfrac{d}{2} - \alpha_1 - \alpha_2\right)^+. \tag{2.17}$$

Then, α_i's ($i = 1,2,3$) satisfy conditions for Lemma 2.2.2. Let also $v, w \in V_{2,\alpha_1 \vee \alpha_2}$. In view of (2.12), we think of $(v \cdot \nabla)w$ as the following linear functional on $\mathcal{V}$:

$$\varphi \mapsto \langle \varphi, (v \cdot \nabla)w \rangle \stackrel{\text{def.}}{=} -\langle w, (v \cdot \nabla)\varphi \rangle,$$

which, by (2.16), extends continuously on $V_{2,1+\alpha_3}$. This way, we regard

$$\begin{aligned} &(v \cdot \nabla)w \in V_{2,-1-\alpha_3}, \\ &\text{with } \|(v \cdot \nabla)w\|_{2,-1-\alpha_3} \le C\sqrt{\|v\|_{2,\alpha_1}\|v\|_{2,\alpha_2}\|w\|_{2,\alpha_1}\|w\|_{2,\alpha_2}}. \end{aligned} \tag{2.18}$$

Let us consider the case $v = w$ and $\alpha_1 \ge \alpha_2$ (Although v and w are identical, it is convenient to take $\alpha_1 > \alpha_2$, as we will see later on). Note that:

$$\Delta v \in V_{2,\alpha_1 - 2} \text{ with } \|\Delta v\|_{2,\alpha_1-2} \le \|v\|_{2,\alpha_1},$$

By this and (2.18), we have that:

$$\begin{aligned} &b(v) \stackrel{\text{def}}{=} \nu\Delta v - (v \cdot \nabla)v \in V_{2,-\beta(\alpha_1,\alpha_2)}, \\ &\text{with } \|b(v)\|_{2,-\beta(\alpha_1,\alpha_2)} \le \nu\|v\|_{2,\alpha_1} + C\|v\|_{2,\alpha_1}\|v\|_{2,\alpha_2}, \end{aligned} \tag{2.19}$$

where

$$\beta(\alpha_1, \alpha_2) = \left(1 + (\tfrac{d}{2} - \alpha_1 - \alpha_2)^+\right) \vee (2 - \alpha_1). \tag{2.20}$$

With this notation, (2.13) takes the form:

$$\langle \varphi, u_t \rangle = \langle \varphi, u_0 \rangle + \int_0^t \langle \varphi, b(u_s) \rangle ds + \int_0^t \langle \varphi, F_s \rangle ds.$$

i.e.,

$$u_t = u_0 + \int_0^t b(u_s)ds + \int_0^t F_s ds \tag{2.21}$$

as linear functionals on $\mathcal{V}$.

Lemma 2.2.3 *Let $\alpha_1 > 0$ and $\alpha_1 \ge \alpha_2 \ge 0$ for which $\beta(\alpha_1, \alpha_2)$ is defined by (2.20). Then, there exists $C \in (0, \infty)$ such that:*

$$\int_0^T \|b(v_t)\|^q_{2,-\beta(\alpha_1,\alpha_2)} dt \le \int_0^T (\nu + C\|v_t\|_{2,\alpha_2})^q \|v_t\|^q_{2,\alpha_1} dt \tag{2.22}$$

for any measurable $v : [0,T] \to V_{2,\alpha_1}$ and $q \in [1, \infty)$. Moreover, for $\alpha > 0$, the following map is continuous:

$$v_\cdot \mapsto \int_0^\cdot b(v_s)ds; \quad L_2([0,T] \to V_{2,\alpha}) \longrightarrow C([0,T] \to V_{2,-\beta(\alpha,\alpha)})$$

Proof: (2.22) is a direct consequence of (2.19). For the rest of this proof, we write $\beta = \beta(\alpha, \alpha)$ for simplicity. Let $v, w \in L_2([0,T] \to V_{2,\alpha})$. Then,

1) $$\sup_{0 \le t \le T} \left\| \int_0^t (b(v_s) - b(w_s))ds \right\|_{2,-\beta} \le \int_0^T \|b(v_s) - b(w_s)\|_{2,-\beta} ds.$$

On the other hand, for $\varphi \in V_{2,-\beta}$,

$$
\begin{aligned}
\langle \varphi, b(v_s) - b(w_s) \rangle &\overset{(2.19)}{=} \underbrace{\nu \langle \Delta\varphi, v_s - w_s \rangle}_{(2)} \underbrace{- \langle v_s, (v_s \cdot \nabla)\varphi \rangle + \langle w_s, (w_s \cdot \nabla)\varphi \rangle}_{(3)}, \\
|(2)| &\le \|\varphi\|_{2,2-\alpha} \|v_s - w_s\|_{2,\alpha} \le \|\varphi\|_{2,\beta} \|v_s - w_s\|_{2,\alpha}, \\
|(3)| &\le |\langle v_s - w_s, (v_s \cdot \nabla)\varphi \rangle| + |\langle w_s, ((v_s - w_s) \cdot \nabla)\varphi \rangle| \\
&\overset{(2.14)}{\le} C\|v_s - w_s\|_{2,\alpha} \|v_s\|_{2,\alpha} \|\varphi\|_{2,\beta} + C\|v_s - w_s\|_{2,\alpha} \|w_s\|_{2,\alpha} \|\varphi\|_{2,\beta},
\end{aligned}
$$

which implies that:

$$
\|b(v_s) - b(w_s)\|_{2,-\beta} \le (\nu + C\|v_s\|_{2,\alpha} + C\|w_s\|_{2,\alpha}) \|v_s - w_s\|_{2,\alpha}.
$$

Plugging this into 1), we arrive at:

$$
\begin{aligned}
&\sup_{0 \le t \le T} \left\| \int_0^t (b(v_s) - b(w_s)) ds \right\|_{2,-\beta} \\
\le\ & \sqrt{3} \left(\int_0^T (\nu^2 + C^2\|v_s\|_{2,\alpha}^2 + C^2\|w_s\|_{2,\alpha}^2) ds \right)^{1/2} \left(\int_0^T \|v_s - w_s\|_{2,\alpha}^2 ds \right)^{1/2},
\end{aligned}
$$

which implies the desired continuity. □

Remark: By (2.22) for $q = 1$ and $(\alpha_1, \alpha_2) = (1, 1)$, we see that

$$
v \in L_2([0,T] \to V_{2,1}) \implies b(v.) \in L_1([0,T] \to V_{2,-\beta(1,1)}) \tag{2.23}
$$

On the other hand, by (2.22) for $q = 2$ and $(\alpha_1, \alpha_2) = (1, 0)$, we see that

$$
v \in L_2([0,T] \to V_{2,1}) \cap L_\infty([0,T] \to V_{2,0}) \implies b(v.) \in L_2([0,T] \to V_{2,-\beta(1,0)}). \tag{2.24}
$$

Note also that:

$$
\beta(1,1) = \begin{cases} 1 & \text{if } d \le 4, \\ \frac{d}{2} - 1 & \text{if } d \ge 5 \end{cases}, \quad \beta(1,0) = \begin{cases} 1 & \text{if } d = 2, \\ \frac{d}{2} & \text{if } d \ge 3 \end{cases}. \tag{2.25}
$$

3 The stochastic Navier-Stokes equation

The construction of a weak solution to the Navier-Stokes equation (2.1)–(2.2) goes back to classical results by J. Leray [Le33, Le34a, Le34b] and E. Hopf [Ho50]. Here, following [Fl08], we consider the case in which the external force is given by a colored noise.

3.1 Introduction of the noise

Throughout this subsection, let H be a separable Hilbert space, and $\Gamma : H \to H$ be a bounded self-adjoint, non-negative definite operator. We suppose in addition that Γ is of *trace class*, that is, the following summation converges for any CONS $\{\varphi_n\}_{n \ge 1}$ of H:

$$
\mathrm{tr}(\Gamma) \overset{\text{def}}{=} \sum_{n \ge 1} \langle \varphi_n, \Gamma\varphi_n \rangle. \tag{3.1}
$$

The number defined above is called the *trace* of Γ and is independent of the choice of the CONS [RS72, p.206, Theorem VI.18].

Definition 3.1.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space.

a) A r.v. $B = (B_t)_{t\geq 0}$ with values in $C([0,\infty) \to \mathbb{R}^d)$ is called a **standard d-dimensional Brownian motion** (abbreviated by BM^d below) if, for each $\theta \in \mathbb{R}^d$ and $0 \leq s < t$,

$$E\left[\exp\left(\mathbf{i}\theta \cdot (B_t - B_s)\right) | \mathcal{G}_s^B\right] = \exp\left(-\frac{t-s}{2}|\theta|^2\right), \quad \text{a.s.} \tag{3.2}$$

where $\mathcal{G}_s^B$ denotes the σ-field generated by $(B_u)_{u\leq s}$. (cf. the complement at the end of this subsection for a definition of the conditional expectation.)

b) A r.v. $W = (W_t)_{t\geq 0}$ with values in $C([0,\infty) \to H)$ is called a **H-valued Brownian motion** with the covariance operator Γ (abbreviated by $\mathrm{BM}(H,\Gamma)$ below) if, for each $\varphi \in H$ and $0 \leq s < t$,

$$E\left[\exp\left(\mathbf{i}\langle \varphi, W_t - W_s \rangle\right) | \mathcal{G}_s^W\right] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\varphi \rangle\right), \quad \text{a.s.} \tag{3.3}$$

where $\mathcal{G}_s^W$ denotes the σ-field generated by $(W_u)_{u\leq s}$.

Remark: The distributional time derivative $\partial_t W_t$ of a $\mathrm{BM}(H,\Gamma)$ $W_t$is sometimes called the *colored noise*.

Exercise 3.1.1 Let W_t be as in Definition 3.1.1 b) and $H_0 \subset H$ be a d-dimensional subspace of H such that $\Gamma H_0 \subset H_0$ with the orthogonal projection π_0. Then, conclude from (3.3) that

$$(\pi_0 W_t)_{t\geq 0} \text{ and } (\sigma B_t)_{t\geq 0} \text{ have the same law,}$$

where $(B_t)_{t\geq 0}$ is BM^d on H_0 (identified with $\mathbb{R}^d$) and $\sigma : H_0 \to H_0$ is a square root of $\Gamma|_{H_0}$. In particular, for each $\varphi \in H$, the process $\langle \varphi, W_t \rangle$, $t \geq 0$ is of the following form:

$$\langle \varphi, W_t \rangle = \sqrt{\langle \varphi, \Gamma\varphi \rangle} B_t, \quad t \geq 0,$$

where $B.$ is a BM^1.

Complement: Let $X \in L_1(P)$ and $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. We define the *conditional expectation* $E[X|\mathcal{G}]$ of X, given $\mathcal{G}$. An implicit definition is given by declaring that $Y = E[X|\mathcal{G}]$ is the unique $\mathcal{G}$-measurable r.v. in $L^1(P)$ such that:

1) $\quad E[Y\mathbf{1}_G] = E[X\mathbf{1}_G]$ for any $G \in \mathcal{G}$.

Another definition is given by explicitly writing down $E[X|\mathcal{G}]$ as a certain Radon Nikodym derivative, which proves that the r.v. Y as referred to above does exist. To do so, we introduce the following signed measure:

$$E^X(F) \stackrel{\text{def}}{=} E[X\mathbf{1}_F], \quad F \in \mathcal{F}.$$

Since $E^X|_{\mathcal{G}}$ is absolutely continuous with respect to $P|_{\mathcal{G}}$, we can define:

$$E[X|\mathcal{G}] = \frac{dE^X|_{\mathcal{G}}}{dP|_{\mathcal{G}}},$$

where the RHS stands for the Radon Nikodym derivative. Then, it is clear that $Y = E[X|\mathcal{G}]$ satisfies 1).

Let us relate the above abstract definition with the elementary conditional expectation of $X \in L_1(P)$, given an event $A \in \mathcal{F}$ with $0 < P(A) < 1$:

$$E[X|A] = \frac{E[X\mathbf{1}_A]}{P(A)}.$$

For the σ-field $\mathcal{G} = \{A, A^c, \emptyset, \Omega\}$, it is clear that

$$E[X|\mathcal{G}] = E[X|A]\mathbf{1}_A + E[X|A^c]\mathbf{1}_{A^c}.$$

3.2 The existence theorem for the stochastic Navier-Stokes equation

We recall (2.19)–(2.21).

Theorem 3.2.1 *Let*

► $\Gamma : V_{2,0} \to V_{2,0}$ *be a self-adjoint, non-negative definite operator of trace class,* $\Delta\Gamma = \Gamma\Delta$ *and;*

► μ_0 *be a Borel probability measure on* $V_{2,0}$ *such that* $m_0 \stackrel{\text{def}}{=} \int \|v\|_2^2 d\mu_0(v) < \infty$.

Then, there exist a process $(X, Y) = ((X_t, Y_t))_{t\geq 0}$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$, *where*

- $X = (X_t)_{t\geq 0}$ *takes values in*

$$L_{2,\mathrm{loc}}([0,\infty) \to V_{2,1}) \cap L_{\infty,\mathrm{loc}}([0,\infty) \to V_{2,0}) \cap C([0,\infty) \to V_{2,-\beta(1,1)}), \tag{3.4}$$

with $\beta(1,1) = 1$ *for* $d \leq 4$ *and* $\beta(1,1) = \frac{d}{2} - 1$ *for* $d \geq 5$. *cf. (2.25);*

- $Y = (Y_t)_{t\geq 0}$ *is a BM*$(V_{2,0}, \Gamma)$ *(cf. Definition 3.1.1).*

The couple (X, Y) *is a weak solution to the Navier-Stokes equation with the initial law* μ_0 *in the sense that:*

$$P(X_0 \in \cdot) = \mu_0; \tag{3.5}$$

$$Y_{t+\cdot} - Y_t \text{ and } \{\langle \varphi, X_s \rangle \ ; \ s \leq t, \varphi \in \mathcal{V}\} \text{ are independent for any } t \geq 0; \tag{3.6}$$

$$\langle \varphi, X_t \rangle = \langle \varphi, X_0 \rangle + \int_0^t \langle \varphi, b(X_s) \rangle ds + \langle \varphi, Y_t \rangle, \quad \text{for all } \varphi \in \mathcal{V} \text{ and } t \geq 0. \tag{3.7}$$

Moreover, the following a priori bounds hold true: for any $T > 0$,

$$E\left[\|X_T\|_2^2 + 2\nu \int_0^T \|X_t\|_{2,1}^2 dt\right] \leq m_0 + \mathrm{tr}(\Gamma)T, \tag{3.8}$$

$$E\left[\sup_{t\leq T} \|X_t\|_2^2\right] \leq (1+T)C < \infty, \tag{3.9}$$

with $C \in (0, \infty)$ *depending only on* $\mathrm{tr}(\Gamma)$, *and* m_0.

Remark: 1) The integral $\int_0^t \langle \varphi, b(X_s) \rangle ds$ in (3.7) is well defined because of (2.23) (or (2.24)) and (3.4).

2) The bound (3.8) is sometimes referred to as the *energy balance inequality.* The interpretation is that

$$\begin{aligned}
\tfrac{1}{2}\|X_T\|_2^2 &= \text{the kinetic energy,} \\
\nu \int_0^T \|X_t\|_{2,1}^2 dt &= \text{the energy dissipated by the friction,} \\
\tfrac{1}{2}\mathrm{tr}(\Gamma)T &= \text{the energy injected from outside (by the colored noise).}
\end{aligned}$$

Although the validity of the equality is not known in general, the equality does hold at the level of finite dimensional approximation (see (5.10) below).

Theorem 3.2.2 *For $d = 2$, the weak solution in Theorem 3.2.1 is* **pathwise unique** *in the sense: if (X, Y) and $(\widetilde{X}, Y)$ are two solutions on a common probability space $(\Omega, \mathcal{F}, P)$ with a common BM($V_{2,0}, \Gamma$) Y such that $X_0 = \widetilde{X}_0$ a.s., then,*

$$P(X_t = \widetilde{X}_t \text{ for all } t \geq 0) = 1.$$

4 The Itô theory for beginners

In this section, we will explain elements in Itô's stochastic calculus without going much into proofs. In what follows, $(\Omega, \mathcal{F}, P)$ is a probability space and $B = (B_t)_{t \geq 0}$ denotes a BM^r.

4.1 Stochastic integrals with respect to the Brownian motion

We fix some notation and terminology:

► A family $X = (X_t)_{t \geq 0}$ of r.v.'s indexed by $t \geq 0$ (most commonly interpreted as "time") is called a *process*. A process X is said ti be *continuous* if $t \mapsto X_t$ is continuous a.s.

► Let $(\mathcal{F}_t)_{t \geq 0}$ be a family of sub σ-fields which are increasing in $t \geq 0$, as such a *filtration*. We assume that it is right-continuous in the sense that:

$$\bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t, \quad t \geq 0. \tag{4.1}$$

► In general, a process $X = (X_t)_{t \geq 0}$ is said to be $(\mathcal{F}_t)$-*adapted*, if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

► We assume that $B = (B_t)_{t \geq 0}$ is a BM^r with respect to $(\mathcal{F}_t)$, that is, B is $(\mathcal{F}_t)$-adapted and

$$E\left[\exp\left(\mathbf{i}\theta \cdot (B_t - B_s)\right) | \mathcal{F}_s\right] = \exp\left(-\frac{t-s}{2}|\theta|^2\right), \quad \text{a.s.} \tag{4.2}$$

for each $\theta \in \mathbb{R}^r$ and $0 \leq s < t$. We also assume that

$$\mathcal{N}^B \subset \mathcal{F}_t, \quad t \geq 0, \tag{4.3}$$

where $\mathcal{N}^B$ is the null-set with respect to B define as follows:

$$\begin{aligned} \mathcal{G}_t^B &= \sigma(B_s,\ s \leq t), \ \ 0 \leq t < \infty, \quad \mathcal{G}_\infty^B = \sigma\left(\cup_{t \geq 0} \mathcal{G}_t^B\right), \\ \mathcal{N}^B &= \{N \subset \Omega, \ ; \ \exists \widetilde{N} \in \mathcal{G}_\infty^B, \ N \subset \widetilde{N}, \ P(\widetilde{N}) = 0\}, \end{aligned}$$

An example of such $(\mathcal{F}_t)_{t \geq 0}$ is given by the *argumented filtration* defined by:

$$\mathcal{F}_t = \sigma\left(\mathcal{G}_t^B \cup \mathcal{N}^B\right). \ \ 0 \leq t < \infty. \tag{4.4}$$

See [KS91,pp.90–91] for the proof the properties (4.1)–(4.2) of the argmented filtration. On the other hand, $\mathcal{G}_t^B$ is *not* right-continuous [KS91,p.89, Problem 7.1].

Definition 4.1.1 (**Stopping times**) A r.v. $\tau : \Omega \to [0, \infty]$ is called a *stopping time* if

$$\{\tau \le t\} \in \mathcal{F}_t \text{ for all } t \ge 0. \tag{4.5}$$

Example 4.1.2 Let $\Gamma \subset \mathbb{R}^r$ and define

$$\tau(\Gamma) = \inf\{t > 0 \ ; \ B_t \in \Gamma\}.$$

It is known that $\tau(\Gamma)$ is a stopping time if $\Gamma \subset \mathbb{R}^r$ is a Borel set. This is not difficult to prove if Γ is either open or closed. Here, in the proof, one sees how the right continuity of $\mathcal{F}_t$ is used.

Consider the following condition[2] for a r.v. $\tau : \Omega \to [0, \infty]$;

$$\{\tau < t\} \in \mathcal{F}_t \text{ for all } t \ge 0. \tag{4.6}$$

Then, this is equivalent to (4.5). In fact, we have

1) $\{\tau < t\} = \cup_{n \ge 1}\{\tau \le t - \frac{1}{n}\}$,

2) $\{\tau > t\} = \cap_{m \ge 1} \cup_{n \ge m} \{\tau \ge t - \frac{1}{n}\}$.

We see from 1) that (4.5) implies (4.6), while the converse can be seen from 2) and the right continuity of $\mathcal{F}_t$.

The observation above can be used to prove that $\tau(\Gamma)$ defined in Example 4.1.2 is a stopping time for an open set Γ. We prove that $\tau(\Gamma)$ satisfies (4.6) as follows:

$$\{\tau(\Gamma) < t\} = \bigcup_{s \in (0,t)} \{B_s \in \Gamma\} = \bigcup_{s \in \mathbb{Q} \cap (0,t)} \{B_s \in \Gamma\} \in \mathcal{F}_t,$$

where, to get the second equality, we have used that Γ is open and that $s \mapsto B_s$ is continuous.

Exercise 4.1.1 Prove that $\tau(\Gamma)$ defined in Example 4.1.2 is a stopping time if Γ is closed. Hint: There is a sequence of open sets $G_1 \supset G_2 \supset \ldots$ such that $\Gamma = \cap_{m \ge 1} G_m$.

We now define some classes of integrands for the stochastic integral.

Definition 4.1.3 (**Integrands for stochastic integral**) We define a function space Φ as the totality of $\varphi : [0, \infty) \times \Omega \to \mathbb{R}$ $((s, \omega) \mapsto \varphi_s(\omega))$ such that[3]:

$$\varphi|_{[0,t] \times \Omega} \text{ is } \mathcal{B}([0,t]) \otimes \mathcal{F}_t \text{ measurable for all } t \ge 0.$$

We also define

$$\begin{aligned} \Phi_2 &= \{\varphi \in \Phi \ ; \ E \textstyle\int_0^t |\varphi_s|^2 ds < \infty \text{ for all } t > 0\}, && (4.7) \\ \Phi_2^{\text{loc.}} &= \{\varphi \in \Phi \ ; \ \textstyle\int_0^t |\varphi_s|^2 ds < \infty, \ P\text{-a.s. for all } t > 0\}. && (4.8) \end{aligned}$$

Clearly, $\Phi_2 \subset \Phi_2^{\text{loc.}} \subset \Phi$.

---

[2]A r.v. τ with this condition is called an *optional time.* We see from the argument of this remark that a stopping time is always an optional time, and that the converse is true when the filtration is right continuous.

[3]This property is called *progressive measurability*

Example 4.1.4 Let $g : \mathbb{R}^r \to \mathbb{R}$ be Borel measurable and

$$\varphi_s(\omega) = g(B_s(\omega)).$$

Then,

- If g is bounded, then $\varphi \in \Phi_2$.
- If $\sup_K |g| < \infty$ for any bounded set $K \subset \mathbb{R}^r$ (in particular, if $g \in C(\mathbb{R}^r)$), then $\varphi \in \Phi_2^{\mathrm{loc.}}$.

Theorem 4.1.5 *For $\varphi \in \Phi_2^{\mathrm{loc.}}$, there are continuous processes (called the* **stochastic integral** *with respect to the Brownian motion)*

$$\left(\int_0^t \varphi_s dB_s^i\right)_{t\geq 0} \quad i = 1, .., r \tag{4.9}$$

with the following properties;

a) *If*

$$\varphi_s(\omega) = \xi_a(\omega) 1_{(a,b]}(s) \tag{4.10}$$

where $0 \leq a < b$ and ξ_a is a bounded, $\mathcal{F}_a$-measurable r.v., then

$$\int_0^t \varphi_s dB_s^i = \xi_a(\omega)(B_{t\wedge b}^i - B_{t\wedge a}^i). \tag{4.11}$$

b) *For $t \geq 0$, $\alpha, \beta \in \mathbb{R}$ and $\varphi, \psi \in \Phi_2^{\mathrm{loc.}}$*

$$\int_0^t (\alpha\varphi_s + \beta\psi_s) dB_s^i = \alpha \int_0^t \varphi_s dB_s^i + \beta \int_0^t \psi_s dB_s^i, \tag{4.12}$$

c) *If $\varphi, \psi \in \Phi_2$ and $t \geq 0$, then,*

$$E\left[\left(\int_0^t \varphi_s dB_s^i\right)\left(\int_0^t \psi_s dB_s^j\right)\right] = \delta_{ij} E \int_0^t \varphi_s \psi_s ds < \infty, \tag{4.13}$$

$$E\left[\int_0^t \varphi_u dB_u^i \middle| \mathcal{F}_s\right] = \int_0^s \varphi_u dB_u^i \quad \textit{whenever } 0 \leq s \leq t. \tag{4.14}$$

We now indicate how the construction of the integrals (4.9) goes (See [KS91, Section 3.2] for details).
Step 1: Let Φ_0 be the set of linear combinations of r.v.'s of the form (4.10). We proceed as follows:

1) For $\varphi \in \Phi_0$, define the integral (4.9) by (4.11) and (4.12).

2) Properties (4.13)–(4.14) hold for $\varphi, \psi \in \Phi_0$ (not difficult to see).

Step 2: We define the integral (4.9) for $\varphi \in \Phi_2$. To do so, we note that Φ_2 is a Fréchet space generated by the semi-norms:

$$\left(E \int_0^T |\varphi_s|^2 ds\right)^{1/2}, \quad T = 1, 2, \ldots$$

We also introduce:

Definition 4.1.6 A process $M = (M_t)_{t \geq 0}$ is said to be a *martingale*, if:

$$\begin{aligned} &(\mathcal{F}_t)\text{-adapted, } M_t \in L_1(P) \text{ for all } t \geq 0; \\ &E[M_t | \mathcal{F}_s] = M_s \ \text{ whenever } 0 \leq s < t. \end{aligned} \tag{4.15}$$

A martingale M is said to be *square integrable*, if $E[M_T^2] < \infty$ for all $T > 0$.

Let

$$\mathcal{M}_2 = \text{the set of continuous, square-integrable martingales.}$$

Then, $\mathcal{M}_2$ is a a Fréchet space generated by the semi-norms:

$$E\left[\sup_{s \leq T} M_s^2\right]^{1/2}, \quad T = 1, 2, ...$$

(cf. (4.16) below). We define:

$$I(\varphi)_t = \int_0^t \varphi_s dB_s^i, \quad \varphi \in \Phi_0, \ \ t \geq 0.$$

We make the following observations:

1) From what we saw in Step 1.2,

$$E[I(\varphi)_T^2] = E \int_0^T |\varphi_s|^2 ds, \ \ I(\varphi) \in \mathcal{M}_2, \ \ \text{for } \varphi \in \Phi_0$$

2) Φ_0 is dense in Φ_2 (cf. [IW89, p.46, Lemma 1.1]). Thus, by 1) above, I extends uniquely to a uniformly continuous mapping $I : \Phi_2 \to \mathcal{M}_2$. This justifies the definition of the integral (4.9) for $\varphi \in \Phi_2$:

$$\int_0^t \varphi_s dB_s^i \stackrel{\text{def.}}{=} I(\varphi)_t, \quad t \geq 0.$$

Properties (4.12)–(4.14) for $\varphi \in \Phi_2$ is then automatic from the construction.

Step 3: We define the integral (4.9) for $\varphi \in \Phi_2^{\text{loc}}$. For $\varphi \in \Phi_2^{\text{loc.}}$, we consider

$$\begin{aligned} \tau^{(n)} &= n \wedge \inf \left\{ t > 0 \, ; \, \int_0^t |\varphi_s|^2 ds \geq n \right\} \\ \varphi_s^{(n)}(\omega) &= \varphi_s(\omega) 1_{[0, \tau^{(n)}]}(s). \end{aligned}$$

Then, $\tau^{(n)} \nearrow \infty$ and $\varphi^{(n)} \in \Phi_2$. We then define the integrals (4.9) by

$$\int_0^t \varphi_s dB_s^i = \int_0^t \varphi_s^{(n)} dB_s^i. \ \ \text{for } t \leq \tau^{(n)}.$$

This finishes the construction.

Finally, we mention the following useful inequality:

Theorem 4.1.7 (Doob's L^2-maximal inequality) *For a square-integrable martingale M,*

$$E\left[\sup_{0\le s\le t} M_s^2\right] \le 4E[M_t^2]. \tag{4.16}$$

In particular, if $\varphi \in \Phi_2$, then

$$E\left[\sup_{0\le s\le t}\left|\int_0^s \varphi_u dB_u^i\right|^2\right] \le 4E\int_0^t |\varphi_s|^2 ds. \tag{4.17}$$

For a proof, see e.g.[IW89, p.33, Theorem 6.10], [KS91, p.13, 3.8 Theorem].

4.2 Itô's formula for semi-martingales

Definition 4.2.1 Let $(\mathcal{F}_t)$ be a right-continuous filtration and $B = (B_t)_{t\ge 0}$ be a BM^r with respect to $(\mathcal{F}_t)$ (cf. (4.1)–(4.3)).
▶ An $\mathbb{R}^d$-valued process $X = (X_t)_{t\ge 0}$ is said to be a *semi-martingale*[4] if it is of the following form:

$$X_t = X_0 + \int_0^t \sigma_s dB_s + \int_0^t b_s ds, \tag{4.18}$$

or more precisely,

$$X_t^i = X_0^i + \sum_{j=1}^r \int_0^t \sigma_s^{ij} dB_s^j + \int_0^t b_s^i, \quad i = 1, ..., d.$$

where

- X_0 is a $\mathcal{F}_0$-measurable r.v.;
- $\sigma = (\sigma^{ij})$ is a matrix with $\sigma^{ij} \in \Phi_2^{\rm loc}$ (cf. (4.8));
- $b = (b_t)_{t\ge 0}$ is an $(\mathcal{F}_t)$-adapted process such that $t \mapsto b_t$ is continuous.

▶ For the semi-martingale (4.18) and a process $(\varphi_t)_{t\ge 0}$, we define:

$$\int_0^t \varphi_s dX_s^i = \sum_{j=1}^r \int_0^t \varphi_s \sigma_s^{ij} dB_s^j + \int_0^t \varphi_s b_s^i ds, \quad i = 1, ...d, \tag{4.19}$$

if each integral on the RHS is well defined, i.e.,

$$\varphi\sigma^{ij} \in \Phi_2^{\rm loc} \text{ and } \int_0^t |\varphi_s b_s^i| ds < \infty \text{ a.s. } i, j = 1, ..., d.$$

The integral (4.19) is called the *stochastic integral* with respect to the semi-martingale (4.18).
▶ For a semi-martingale (4.18), we define the *bracket processes* by:

$$\langle X^i, X^j \rangle_t = \sum_{k=1}^r \int_0^t \sigma_s^{ik}\sigma_s^{jk} ds, \quad i, j = 1, ..., d. \tag{4.20}$$

[4]Here, we only consider a limited class of what is usually referred to as the "semi-martingale" cf. [IW89, p.64, Definition 4.1]

Theorem 4.2.2 (Itô's formula for semi-martingales) *Suppose that X is a semi-martingale given by (4.18) and $f \in C^2(\mathbb{R}^d)$. Then, P-a.s.,*

$$\begin{aligned} & f(X_t) - f(X_0) \\ = & \sum_{i=1}^d \int_0^t \partial_i f(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_i \partial_j f(X_s) d\langle X^i, Y^j \rangle_s, \quad \textit{for all } t \geq 0. \end{aligned} \tag{4.21}$$

The proof goes along the following line (e.g.[IW89, pp.67–71], [KS91, pp.150–153]). Let $d = r = 1$ for simplicity, and $0 = t_0 < t_1 < \ldots < t_n = t$ be the division for which $\delta_n \stackrel{\text{def}}{=} \max_{1 \leq k \leq n}(t_k - t_{k-1}) \to 0$ $(n \to \infty)$. For the indices to be read easily, we write $\widetilde{X}_k = X_{t_k}$. Then, by Taylor expanding f around $\widetilde{X}_{k-1}$, we have:

$$f(\widetilde{X}_k) - f(\widetilde{X}_{k-1}) = f'(\widetilde{X}_{k-1})\Delta_k + \tfrac{1}{2} f''(\widetilde{X}_{k-1} + \theta_k \Delta_k)\Delta_k^2$$

where $\Delta_k = \widetilde{X}_k - \widetilde{X}_{k-1}$ and $\theta_k \in (0,1)$. This implies that:

$$f(X_t) - f(X_0) = \underbrace{\sum_{k=1}^n f'(\widetilde{X}_{k-1})\Delta_k}_{=:I_n} + \tfrac{1}{2} \underbrace{\sum_{k=1}^n f''(\widetilde{X}_{k-1} + \theta_k \Delta_k)\Delta_k^2}_{=:J_n}.$$

By verifying

$$\lim_{n \to \infty} I_n = \int_0^t f'(X_s) dX_s \quad \text{and} \quad \lim_{n \to \infty} J_n = \int_0^t f''(X_s) d\langle X, X \rangle_s,$$

in an appropriate sense, one obtains (4.21) for $d = r = 1$. The extension to general d, r is straightforward.

Example 4.2.3 For the semi-martingale (4.18), we have:

$$|X_t|^2 - |X_0|^2 = 2M_t + \int_0^t \left(2X_s \cdot b_s + |\sigma_s|^2\right) ds, \quad \text{with } M_t = \sum_{\substack{1 \leq i \leq d \\ 1 \leq j \leq r}} \int_0^t X_s^i \sigma_s^{ij} dB_s^j. \tag{4.22}$$

Here, and in what follows, $|\sigma|^2 = \sum_{\substack{1 \leq i \leq d \\ 1 \leq j \leq r}} (\sigma^{ij})^2$. Suppose in particular that

$$E[|X_0|^2] \leq m_0 < \infty, \quad X_t \cdot b_t \leq C, \quad |\sigma_t|^2 \leq C, \tag{4.23}$$

where m_0 and C is a non-random constant. Then, for any $t > 0$,

$$E\left[|X_t|^2\right] = E[|X_0|^2] + E\int_0^t \left(2X_s \cdot b_s + |\sigma_s|^2\right) ds, \tag{4.24}$$

$$E\left[\sup_{s \leq t} |X_s|^2\right] \leq E[|X_0|^2] + C't, \tag{4.25}$$

where the constant C' depends only on m_0 and C.

Proof: Note that
$$\partial_i |x|^2 = 2x^i, \quad \partial_i \partial_j |x|^2 = 2\delta_{i,j}.$$
Thus, we see from Itô's formula that:
$$|X_t|^2 - |X_0|^2 = \underbrace{\sum_{j=1}^d \int_0^t 2X_s^j \cdot dX_s^j}_{=:I} + \underbrace{\tfrac{1}{2}\sum_{i,j=1}^d \int_0^t 2\delta_{i,j} d\langle X^i, X^j \rangle_s}_{=:J},$$
with
$$\begin{aligned} I &= 2M_t + 2\int_0^t X_s \cdot b(X_s) ds, \\ J &= \sum_{1 \le i \le d} \langle X^i, X^i \rangle_t \stackrel{(4.20)}{=} \int_0^t \underbrace{\sum_{i,k=1}^d (\sigma_s^{ik})^2}_{=|\sigma_s|^2} ds. \end{aligned}$$
This proves (4.22). We next assume (4.23) to show (4.24)–(4.25). This will be straightforward, once we know that $M.$ is a square-integrable martingale. However, we have to settle this technical point first. We start by showing that:

1) $\quad E[|X_t|^2] \le m_0 + 3Ct,$

Since $X.$ is continuous and $|X_0| < \infty$ a.s., we have that:
$$e_n \stackrel{\text{def}}{=} \inf\{t \, ; \, |X_t| \ge n\} \nearrow \infty, \quad \text{as } n \nearrow \infty.$$
Note also that:
$$M_{t \wedge e_n} = \sum_{\substack{1 \le i \le d \\ 1 \le j \le r}} \int_0^{t \wedge e_n} X_s^i \sigma_s^{ij} dB_s^j = \sum_{\substack{1 \le i \le d \\ 1 \le j \le r}} \int_0^t \mathbf{1}_{\{s \le e_n\}} X_s^i \sigma_s^{ij} dB_s^j$$
and that $\mathbf{1}_{\{s \le e_n\}} X_s^i \sigma_s^{ij} \in \Phi_2$. These and (4.14) imply that $E[M_{t \wedge e_n}] = 0$. Combining this with:

2) $\quad |X_t|^2 \stackrel{(4.22),\,(4.23)}{\le} |X_0|^2 + 2M_t + 3Ct,$

we have that:
$$E[X_{t \wedge e_n}^2] \le m_0 + 3Ct.$$
Thus, 1) follows from Fatou's lemma. 1) and (4.23) imply that:
$$X_s^i \sigma_s^{ij} \in \Phi_2.$$
Then, $E[M_t] = 0$ by (4.14). Thus, (4.24) follows from (4.22) taking expectation. We next show that

3) $\quad E\left[\sup_{s \le t} |M_s|^2\right] \le C_1(t + t^2).$

To do so, we start by noting that:

4) $\sum_j (\sum_i X_s^i \sigma_s^{ij})^2 = |\sigma_s^* X_s|^2 \le |\sigma_s|^2 |X_s|^2$.

Then,

$$\begin{aligned} E\left[\sup_{s\le t} |M_s|^2\right] &\overset{(4.16)}{\le} 4E\left[|M_t|^2\right] \overset{(4.13)}{=} 4\sum_j E\int_0^t \left(\sum_i X_s^i \sigma_s^{ij}\right)^2 ds \\ &\overset{4)}{\le} 4E\int_0^t |\sigma_s|^2 |X_s|^2 ds \overset{1),\,(4.23)}{\le} 4C(m_0 t + \tfrac{3C}{2}t^2). \end{aligned}$$

we then get (4.22) as follows:

$$E\left[\sup_{s\le t} |X_s|^2\right] \overset{2)}{\le} m_0 + 2E\left[\sup_{s\le t} |M_s|^2\right]^{1/2} + 3Ct \overset{3)}{\le} m_0 + C_2 t.$$

□

Example 4.2.4 (Itô's formula for the Brownian motion) Suppose that $f \in C^2(\mathbb{R}^r)$. Then, P-a.s.,

$$f(B_t) - f(0) = \sum_{1\le i\le r} \int_0^t \partial_i f(B_s) dB_s^i + \frac{1}{2}\int_0^t \Delta f(B_s) ds, \quad \text{for all } t \ge 0. \tag{4.26}$$

Proof: A special case of (4.21) with $d = r$, $\sigma^{ij} = \delta^{ij}$, and $b \equiv 0$. □

4.3 Stochastic differential equations: an existence and uniqueness theorem

Let $\sigma \in C(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r)$, $b \in C(\mathbb{R}^d \to \mathbb{R}^d)$ and ξ be an $\mathbb{R}^d$-valued r.v. We consider a stochastic differential equation (SDE):

$$X_t = \xi + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds, \tag{4.27}$$

or more precisely,

$$X_t^i = \xi^i + \sum_{j=1}^r \int_0^t \sigma^{ij}(X_s) dB_s^j + \int_0^t b^i(X_s), \quad i = 1, ..., d.$$

We define:

$$\begin{aligned} \mathcal{G}_t^{\xi,B} &= \sigma(\xi, B_s,\ s \le t),\ 0 \le t < \infty,\ \ \mathcal{G}_\infty^{\xi,B} = \sigma\left(\cup_{t\ge 0} \mathcal{G}_t^{\xi,B}\right), \\ \mathcal{N}^{\xi,B} &= \{N \subset \Omega,\ ;\ \exists \widetilde{N} \in \mathcal{G}_\infty^{\xi,B},\ N \subset \widetilde{N},\ P(\widetilde{N}) = 0\}, \end{aligned}$$

and

$$\mathcal{F}_t^{\xi,B} = \sigma\left(\mathcal{G}_t^{\xi,B} \cup \mathcal{N}^{\xi,B}\right), \quad 0 \le t < \infty. \tag{4.28}$$

We now state the following existence and uniqueness theorem:

Theorem 4.3.1 *Referring to (4.27), suppose that*

$$m_0 \stackrel{\text{def}}{=} E[|\xi|^2] < \infty$$

and that there exist $K, L_n \in (0,\infty)$, $n = 1,2,\ldots$ *such that:*

$$\begin{aligned} |\sigma(x)-\sigma(y)|^2 + |b(x)-b(y)|^2 &\leq L_n|x-y|^2 \quad \text{if } |x|,|y| \leq n, & (4.29)\\ |\sigma(x)|^2 + 2x\cdot b(x) &\leq K(1+|x|^2), \quad x \in \mathbb{R}^d. & (4.30)\end{aligned}$$

Then, there exists a unique process $X.$ *such that:*

a) X_t *is* $\mathcal{F}_t^{\xi,B}$*-measurable for all* $t \geq 0$ *(cf. (4.28));*

b) *the SDE (4.27) is satisfied.*

Proof: By [IW89, p.178, Theorem 3.1], the condition (4.29) ensures existence of the unique solution admitting the possibility of explosion at finite time:

$$\lim_{t\nearrow\tau} |X_t| = \infty, \quad \text{for some } \tau < \infty.$$

However, such possibility is excluded by the condition (4.30) [IW89, p.177, Theorem 2.4]. □

5 The Galerkin approximation

5.1 The approximating SDE

For each $z \in \mathbb{Z}^d\backslash\{0\}$, let $\{e_{z,j}\}_{j=1}^{d-1} \subset \mathbb{R}^d$ be an orthonormal basis of the hyperplane:

$$\{x \in \mathbb{R}^d \; ; \; z\cdot x = 0\}$$

and let:

$$\psi_{z,j}(x) = \left\{ \begin{array}{ll} \sqrt{2}e_{z,j}\cos(2\pi z\cdot x), & j = 1,\ldots,d-1, \\ \sqrt{2}e_{z,|j|}\sin(2\pi z\cdot x), & j = -1,\ldots,-(d-1) \end{array}\right. , \quad x \in \mathbb{T}^d. \tag{5.1}$$

Then,

$$\{\psi_{z,j} \; ; \; z \in \mathbb{Z}^d\backslash\{0\}, \; j = \pm 1,\ldots,\pm(d-1)\}$$

is an orthonormal basis of $V_{2,0}$. We also introduce:

$$\begin{array}{rcl} \mathcal{V}_n &=& \text{the linear span of } \{\psi_{z,j} \; ; \; (z,j) \text{ with } z \in [-n,n]^d\}, \\ \mathcal{P}_n &=& \text{the orthogonal projection} : L^2(\mathbb{T}^d \to \mathbb{R}^d) \to \mathcal{V}_n. \end{array} \tag{5.2}$$

Using the orthonormal basis (5.1), we identify $\mathcal{V}_n$ with $\mathbb{R}^N$, $N = \dim \mathcal{V}_n$. Let μ_0 and $\Gamma : V_{2,0} \to V_{2,0}$ be as in Theorem 3.2.1. Let also ξ be a r.v. such that $P(\xi \in \cdot) = \mu_0$. Finally, let W_t be a BM(V_0,Γ) defined on a probability space $(\Omega,\mathcal{F},P)$. Then, $\mathcal{P}_nW_t$ is identified with an N-dimensional Brownian motion with covariance matrix $\Gamma\mathcal{P}_n$. Then, we consider the following approximation of (3.7):

$$X_t^n = X_0^n + \int_0^t \mathcal{P}_n b(X_s^n)ds + \mathcal{P}_nW_t \quad t \geq 0, \tag{5.3}$$

where $X_0^n = \mathcal{P}_n \xi$. Let:

$$X_t^{n,z,j} = \langle \psi_{z,j}, X_t^n \rangle \text{ and } W_t^{z,j} = \langle \psi_{z,j}, W_t \rangle \tag{5.4}$$

be the (z,j)-coordinates of X_t^n and W_t. Then, (5.3) reads:

$$X_t^{n,z,j} = X_0^{n,z,j} + \int_0^t b^{z,j}(X_s^n)ds + W_t^{z,j}, \tag{5.5}$$

where

$$b^{z,j}(v) = \langle v, (v \cdot \nabla)\psi_{z,j} \rangle + \nu \langle v, \Delta \psi_{z,j} \rangle, \quad v \in \mathcal{V}_n. \tag{5.6}$$

Let $\gamma_{z,j} \geq 0$ be such that $\Gamma \psi_{z,j} = \gamma_{z,j} \psi_{z,j}$ and $I_n = \{(z,j) \; ; \; |z| \leq n, \; \gamma_{z,j} > 0\}$. Then,

$$B_\cdot^{z,j} = \frac{W_t^{z,j}}{\sqrt{\gamma_{z,j}}}, \quad (z,j) \in I_n$$

are independent BM^1's and

$$\mathcal{P}_n W_t = \sum_{(z,j) \in I_n} W_t^{z,j} \psi_{z,j} = \sum_{(z,j) \in I_n} \sqrt{\gamma_{z,j}} B_t^{z,j} \psi_{z,j}.$$

Thus, the SDE (5.3) can be thought of as a special case of (4.27), where

$$\sigma(\cdot) \text{ is a constant diagonal matrix with } |\sigma(\cdot)|^2 = \mathrm{tr}(\Gamma \mathcal{P}_n). \tag{5.7}$$

Also by (5.6),

$$\text{the drift } \mathcal{P}_n b(v) \text{ is a polynomial in } v \in \mathcal{V}_n \text{ of degree two.} \tag{5.8}$$

Moreover, for $v \in \mathcal{V}_n$,

$$\langle v, \mathcal{P}_n b(v) \rangle = \langle v, \nu \Delta v + (v \cdot \nabla) v \rangle \stackrel{\text{Lemma 2.1.2}}{=} \nu \langle v, \Delta v \rangle = -\nu \|\nabla v\|_2^2 \leq 0. \tag{5.9}$$

We see from (5.7)–(5.9) above that the SDE (5.3) satisfies the assumptions (4.29)–(4.30) of Theorem 4.3.1, and hence admits a unique solution. The solution is then a semi-martingale of the form (4.18) for which the assumption (4.23) of Example 4.2.3 is valid. Therefore, for any $T > 0$,

$$E\left[\|X_T^n\|_2^2 + 2\nu \int_0^T \|X_t^n\|_{2,1}^2 dt\right] = E[\|X_0^n\|_2^2] + \mathrm{tr}(\Gamma \mathcal{P}_n)T, \tag{5.10}$$

$$E\left[\sup_{t \leq T} \|X_t^n\|_2^2\right] \leq (1+T^2)C < \infty, \tag{5.11}$$

where $C = C(\Gamma, m_0) \in (0, \infty)$.

We will summarize the above considerations as Theorem 5.1.1 below. To do so, we define:

$$\begin{aligned} \mathcal{G}_t^{\xi,W} &= \sigma(\xi, W_s, \; s \leq t), \;\; 0 \leq t < \infty, \;\; \mathcal{G}_\infty^{\xi,W} = \sigma\left(\cup_{t \geq 0} \mathcal{G}_t^{\xi,W}\right), \\ \mathcal{N}^{\xi,W} &= \{N \subset \Omega, \; ; \; \exists \widetilde{N} \in \mathcal{G}_\infty^{\xi,W}, \; N \subset \widetilde{N}, \; P(\widetilde{N}) = 0\}, \end{aligned}$$

and

$$\mathcal{F}_t^{\xi,W} = \sigma\left(\mathcal{G}_t^{\xi,W} \cup \mathcal{N}^{\xi,W}\right), \quad 0 \leq t < \infty. \tag{5.12}$$

Theorem 5.1.1 *Let $W_\cdot$, ξ, and $\mathcal{F}_t^{\xi,W}$ as above. Then, for each n, there exists a unique process $X_\cdot^n$ such that:*

a) *X_t^n is $\mathcal{F}_t^{\xi,W}$-measurable for all $t \geq 0$;*

b) *(5.3), (5.10) and (5.11) are satisfied;*

5.2 Compact imbedding lemmas

We will need some compact imbedding lemmas from [FG95]. We first introduce:

Definition 5.2.1 Let $p \in [1, \infty)$, $T \in (0, \infty)$, and E be a Banach space.

a) We let $L_{p,1}([0,T] \to E)$ denote the Sobolev space of all $u \in L_p([0,T] \to E)$ such that:

$$u(t) = u(0) + \int_0^t u'(s)ds, \quad \text{for almost all } t \in [0,T]$$

with some $u(0) \in E$ and $u'(\cdot) \in L_p([0,T] \to E)$. We endow the space $L_{p,1}([0,T] \to E)$ with the norm $\|u\|_{L_{p,1}([0,T]\to E)}$ defined by

$$\|u\|^p_{L_{p,1}([0,T]\to E)} = \int_0^T (|u(t)|^p_E + |u'(t)|^p_E)dt.$$

b) For $\alpha \in (0,1)$, we let $L_{p,\alpha}([0,T] \to E)$ denote the Sobolev space of all $u \in L_p([0,T] \to E)$ such that:

$$\int_{0<s<t<T} \frac{|u(t) - u(s)|^p_E}{|t-s|^{1+\alpha p}} dsdt < \infty.$$

We endow the space $L_{p,\alpha}([0,T] \to E)$ with the norm $\|u\|_{L_{p,\alpha}([0,T]\to E)}$ defined by

$$\|u\|^p_{L_{p,\alpha}([0,T]\to E)} = \int_0^T |u(t)|^p dt + \int_{0<s<t<T} \frac{|u(t) - u(s)|^p_E}{|t-s|^{1+\alpha p}} dsdt.$$

Remark: Note that:

$$\int_{0<s<t<T} \frac{dsdt}{|t-s|^{1+\lambda}} = \begin{cases} \infty & \text{if } \lambda \geq 0, \\ \frac{T^{1+|\lambda|}}{(1+|\lambda|)|\lambda|} & \text{if } \lambda < 0 \end{cases} \tag{5.13}$$

Therefore, roughly speaking, a function in $L_{p,\alpha}([0,T] \to E)$ is, "Hölder continuous with the exponent bigger than α".

Exercise 5.2.1 Prove that $L_{p,\beta}([0,T] \to E) \hookrightarrow L_{p,\alpha}([0,T] \to E)$ if $0 < \alpha < \beta \leq 1$.

Lemma 5.2.2 *[FG95, p.370, Theorem 2.1] Let:*

► *$E_1, ..., E_n$ and E be Banach spaces such that each $E_i \hookrightarrow\hookrightarrow E$, $i = 1, ..., n$.*

► *$p_1, ..., p_n \in (1, \infty)$, $\alpha_1, ..., \alpha_n \in (0,1)$ are such that $p_i\alpha_i > 1$, $i = 1, ..., n$.*

Then, for any $T > 0$,

$$L_{p_1,\alpha_1}([0,T] \to E_1) + ... + L_{p_n,\alpha_n}([0,T] \to E_n) \hookrightarrow\hookrightarrow C([0,T] \to E).$$

Lemma 5.2.3 *[FG95, p.372, Theorem 2.2] Let:*

$$E_0 \hookrightarrow\hookrightarrow E \hookrightarrow E_1$$

be Banach spaces such that the first imbedding is compact, and E_0, E_1 are reflexible. Then, for any $p \in (1, \infty)$, $\alpha \in (0,1)$ and $T > 0$,

$$L_p([0,T] \to E_0) \cap L_{p,\alpha}([0,T] \to E_1) \hookrightarrow\hookrightarrow L_p([0,T] \to E).$$

5.3 Regularity of the noise

Let H be a separable Hilbert space, and $\Gamma : H \to H$ be a non-negative self-adjoint operator of trace class, as in section 3.1. By the Hilbert-Schmidt theorem [RS72, p.203, Theorem VI.16], there exist a CONS $(\varphi_n)_{n\geq 1}$ of H and numbers $\gamma_n \geq 0$ such that:

$$\Gamma\varphi_n = \gamma_n\varphi_n, \quad n \geq 1. \tag{5.14}$$

Let W be a BM(H,Γ). Then, the processes:

$$B^k_\cdot \stackrel{\text{def}}{=} \langle W_\cdot, \varphi_k \rangle/\sqrt{\gamma_k}, \quad k \in I \stackrel{\text{def}}{=} \{k \in \mathbb{N}\ ;\ \gamma_k > 0\}$$

are independent BM^1's. Let $\{B^k_\cdot\}_{k\in N\setminus I}$ be independent BM^1's which are independent of $\{B^k_\cdot\}_{k\in I}$. Then, $\langle W_\cdot, \varphi_k \rangle = \sqrt{\gamma_k}B^k_\cdot$ for all $k \in \mathbb{N}$, and thus,

$$W_t = \sum_{k=0}^{\infty}\langle W_t, \varphi_k \rangle\varphi_k = \sum_{k=0}^{\infty}\sqrt{\gamma_k}B^k_t\varphi_k, \quad t \geq 0.$$

Let us consider the finite summation:

$$W^n_t = \sum_{k=0}^{n}\langle W_t, \varphi_k \rangle\varphi_k = \sum_{k=0}^{n}\sqrt{\gamma_k}B^k_t\varphi_k, \quad t \geq 0, \tag{5.15}$$

Lemma 5.3.1 *Referring to (5.15), for any $p \in [1,\infty)$, $\alpha \in [0,1/2)$ and $T > 0$, there exists $C = C_{\alpha,p,T} \in (0,\infty)$ such that:*

$$\sup_{n\geq 0} E[\|W^n_\cdot\|^p_{L_{p,\alpha}([0,T]\to H)}] \leq C\mathrm{tr}(\Gamma)^{p/2}. \tag{5.16}$$

Proof: We first prepare an exponential moment bound. Let $\varepsilon \in (0,1)$, $\lambda, t \geq 0$ be such that $0 \leq \lambda t\gamma_k \leq 1-\varepsilon$ for all $k \in \mathbb{N}$. Then,

1) $E\left[\exp\left(\frac{\lambda}{2}\|W^n_t\|^2\right)\right] = \prod_{k=0}^{n}\frac{1}{\sqrt{1-\lambda t\gamma_k}} \leq \exp\left(\frac{\lambda t}{2\varepsilon}\mathrm{tr}(\Gamma)\right).$

Since $\|W^n_t\|^2 = \sum_{k=0}^n \gamma_k|B^k_t|^2$,

$$\begin{aligned}
E\left[\exp\left(\frac{\lambda}{2}\|W^n_t\|^2\right)\right] &= \prod_{k=0}^{n}E\left[\exp\left(\frac{\lambda\gamma_k}{2}|B^k_t|^2\right)\right] \\
&= \prod_{k=0}^{n}\frac{1}{\sqrt{2\pi t}}\underbrace{\int_{\mathbb{R}}\exp\left(-\left(\frac{1}{t}-\lambda\gamma_k\right)\frac{x^2}{2}\right)}_{=\sqrt{\frac{2\pi}{\frac{1}{t}-\lambda\gamma_k}}} = \prod_{k=0}^{n}\frac{1}{\sqrt{1-\lambda t\gamma_k}}.
\end{aligned}$$

We next observe for any $\delta \in [0, 1-\varepsilon]$ that

$$\frac{1}{1-\delta} = 1 + \frac{\delta}{1-\delta} \leq 1 + \frac{\delta}{\varepsilon} \leq e^{\frac{\delta}{\varepsilon}}.$$

Hence, considering $\delta = \lambda t\gamma_k$ and taking the square root, and then the product over $k = 0, .., n$, we have

$$\prod_{k=0}^{n}\frac{1}{\sqrt{1-\lambda t\gamma_k}} \leq \exp\left(\frac{\lambda t}{2\varepsilon}\mathrm{tr}(\Gamma)\right).$$

Thus, we get 1). Then, it is not difficult (Exercise 5.3.1 below) to see from 1) that

2) $E\left[\|W_t^n\|^p\right] \le C_p\,(\mathrm{tr}(\Gamma)t)^{p/2}$ for any $p \in (0,\infty)$,

with $C_p \in (0,\infty)$ depending only on p. Noting that

$$E\left[\|W_t^n - W_s^n\|^p\right] = E\left[\|W_{t-s}^n\|^p\right] \overset{2)}{\le} C_p\,(\mathrm{tr}(\Gamma)(t-s))^{p/2}, \quad s<t,$$

we get

$$\begin{aligned} E\int_{0<s<t<T} \frac{\|W_t^n - W_s^n\|^p}{(t-s)^{1+\alpha p}}dsdt &\le C_p \mathrm{tr}(\Gamma)^{p/2}\int_{0<s<t<T}\frac{dsdt}{(t-s)^{1+(\alpha-\frac{1}{2})p}} \\ &\le C_{p,\alpha}\mathrm{tr}(\Gamma)^{p/2}T^{1+(\frac{1}{2}-\alpha)p}. \end{aligned}$$

This and 2) imply (5.16). □

Exercise 5.3.1 Conclude 2) from 1) in the proof of Lemma 5.3.1. Hint: Take $\lambda = \frac{1}{2\mathrm{tr}(\Gamma)t}$ in 1).

5.4 A digression on tightness

Let $X^n = (X_t^n)_{t\ge 0} \in \mathcal{V}$ be the unique solution of (5.3) for the Galerkin approximation. In section 5.5, we will find a "convergent subsequence", the limit of which eventually solves (3.7). This can be done by showing that the laws of X^n, $n \in \mathbb{N}$ are tight (see Definition 5.4.1). This subsection serves as a collection of notions and facts regarding the tightness, which we will use in section 5.5.

Throughout this subsection, let $S = (S,\rho)$ be a separable metric space and $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 5.4.1 A sequence $\{X_n : \Omega \to S\}_{n\in\mathbb{N}}$ of r.v.'s (or more precisely, the laws of these r.v.'s) are said to be *tight*, if, for any $\varepsilon \in (0,1)$, there exists a relatively compact set $K \subset S$ such that:

$$\inf_{n\in\mathbb{N}} P(X_n \in K) \ge 1-\varepsilon.$$

Here is a common way to check the tightness:

Lemma 5.4.2 *Let $\{X_n : \Omega \to S\}_{n\in\mathbb{N}}$ be r.v.'s. Suppose that there exists a function $F: S \to [0,\infty)$ such that:*

the set $K_R \overset{\text{def}}{=} \{x \in S\ ;\ F(x) \le R\}$ is relatively compact for all $R>0$;
$\sup_{n\in\mathbb{N}} E[F(X_n)] \le C < \infty.$

Then, $\{X_n\}_{n\in\mathbb{N}}$ are tight.

Proof: We then have that:

$$\begin{aligned} \sup_{n\in\mathbb{N}} P(X_n \notin K_R) &= \sup_{n\in\mathbb{N}} P(F(X_n) > R) \\ &\le \sup_{n\in\mathbb{N}} \frac{E[F(X_n)]}{R} \le \frac{C}{R} \to 0. \end{aligned}$$

This proves the tightness. □

Once we are able to check that a sequence of r.v.'s is right, we have the following consequence:

Lemma 5.4.3 *Suppose that S is complete and that a sequence $\{X_n : \Omega \to S\}_{n\in\mathbb{N}}$ of r.v.'s are tight. Then, there exist a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, a sequence $n(k) \nearrow \infty$ of integers, and a sequence*

$$\{\widetilde{X}_k : \widetilde{\Omega} \to S\}_{k\in\mathbb{N}\cup\{\infty\}}$$

of r.v.'s such that:

$$\begin{aligned} \widetilde{P}(\widetilde{X}_k \in \cdot) &= P(X_{n(k)} \in \cdot) \text{ for all } k \in \mathbb{N}; \\ \lim_{k\to\infty} \widetilde{X}_k &= \widetilde{X}_\infty, \ \widetilde{P}\text{-a.s.} \end{aligned}$$

Proof: This is a consequence of Prohorov's theorem [IW89, p.7, Theorem 2.6] and Skorohod's representation theorem [IW89, p.9, Theorem 2.7]. □

Lemma 5.4.4 *Suppose that (S_j, ρ_j) $(j = 1,..,m)$ are complete separable metric spaces such that all of S_j $(j = 1, ..., m)$ are subsets of a common set. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with values in $S \stackrel{\text{def}}{=} \bigcap_{j=1}^m S_j$ which is tight in each of (S_j, ρ_j), $j = 1,..,m$ separately. Then, there exist a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, a sequence $n(k) \nearrow \infty$ of integers, and a sequence*

$$\{\widetilde{X}_k : \widetilde{\Omega} \to S\}_{k\in\mathbb{N}\cup\{\infty\}}$$

of r.v.'s such that:

$$\widetilde{P}(\widetilde{X}_k \in \cdot) = P(X_{n(k)} \in \cdot) \text{ for all } k \in \mathbb{N};$$

$$\lim_{k\to\infty} \sum_{j=1}^m \rho_j(X, \widetilde{X}_k) = 0 \ \text{ a.s.}$$

Proof: By induction, it is enough to consider the case of $m = 2$. Let $\varepsilon > 0$ be arbitrary. Then, for $j = 1, 2$, there exists a compact subset K_j of S_j such that:

$$P(X_n \in K_j) \geq 1 - \varepsilon, \text{ for all } j = 1, 2 \text{ and } n = 1, 2, ...$$

Now, a very simple, but crucial observation is that $K_1 \cap K_2$ is compact in $S_1 \cap S_2$ with respect to the metric $\rho_1 + \rho_2$. Also,

$$P(X_n \in K_1 \cap K_2) \geq 1 - 2\varepsilon, \text{ for all } j = 1, 2 \text{ and } n = 1, 2, ...$$

These imply that (X_n) is tight in $S_1 \cap S_2$ with respect to the metric $\rho_1 + \rho_2$. Thus, the lemma follows from Lemma 5.4.3. □

5.5 Convergence of the approximation along a subsequence

Let $X^n = (X_t^n)_{t\geq 0} \in \mathcal{V}$ be the unique solution of (5.3) for the Galerkin approximation. Recall the notation from (2.25):

$$\beta(1, 0) = \begin{cases} 1 & \text{if } d = 2, \\ \frac{d}{2} & \text{if } d \geq 3 \end{cases}.$$

Proposition 5.5.1 *For $\alpha \in [0, 1)$ and $\beta > \beta(1, 0)$ (cf. (2.25)), Then, there exist a process X and a sequence $(\widetilde{X}^k)_{k\geq 1}$ of processes defined on a probability space $(\Omega, \mathcal{F}, P)$ such that the following properties are satisfied:*

a) *The process X takes values in*

$$C([0,\infty) \to V_{2,-\beta}) \cap L_{2,\mathrm{loc}}([0,\infty) \to V_{2,\alpha}). \tag{5.17}$$

b) *For some sequence $n(k) \nearrow \infty$, $\widetilde{X}^k$ has the same law as $X^{n(k)}$ and*

$$\lim_{k\to\infty} \widetilde{X}^k = X \quad \textit{in the metric space (5.17), P-a.s.} \tag{5.18}$$

We divide the proof of Proposition 5.5.1 into the series of lemmas: To prepare the proof of these lemmas, we write (5.3) as:

$$X_t^n = X_0^n + J_t^n + W_t^n \text{ with } J_t^n = \int_0^t \mathcal{P}_n b(X_s^n)ds. \tag{5.19}$$

Lemma 5.5.2 *Let $\beta(1,0)$ and J_t^n be as in (2.25) and (5.19). Then, there exists $C_T \in (0,\infty)$ such that:*

$$\sup_{n\geq 1} E\left[\|J_\cdot^n\|_{L_{2,1}([0,T]\to V_{2,-\beta(1,0)})}\right] \leq C_T < \infty. \tag{5.20}$$

Proof: It is not difficult to see that:

1) $$\|J_\cdot^n\|^2_{L_{2,1}([0,T]\to V_{2,-\beta(1,0)})} \leq C_T \int_0^T \|\mathcal{P}_n b(X_s^n)\|^2_{V_{2,-\beta(1,0)}} ds. \quad \text{(cf. Exercise 5.5.1)}$$

By (2.22) for $q=2$ and $(\alpha_1,\alpha_2) = (1,0)$, we see that

2) $$\begin{aligned}\int_0^T \|b(X_s^n)\|^2_{2,-\beta(1,0)}dt &\leq \int_0^T (\nu + C\|X_s^n\|_2)^2 \|X_s^n\|^2_{2,1} ds \\ &\leq (\nu + C\sup_{s\leq T}\|X_s^n\|_2)^2 \int_0^T \|X_s^n\|^2_{2,1} ds.\end{aligned}$$

Since $\mathcal{P}_n$ is contraction on $V_{2,\alpha}$ for any $\alpha \in \mathbb{R}$, we can combine the above bounds and (5.10)–(5.11) to obtain n (5.20) as follows:

$$\begin{aligned}E\left[\|J_\cdot^n\|_{L_{2,1}([0,T]\to V_{2,-\beta(1,0)})}\right] &\overset{1)-2)}{\leq} C_T E\left[(\nu + C\sup_{s\leq T}\|X_s^n\|_2)\left(\int_0^T \|X_s^n\|^2_{2,1}ds\right)^{1/2}\right] \\ &\leq C_T E\left[(\nu + C\sup_{s\leq T}\|X_s^n\|_2)^2\right]^{1/2} E\left[\int_0^T \|X_s^n\|^2_{2,1}ds\right]^{1/2} \\ &\overset{(5.10)-(5.11)}{\leq} C_T' < \infty.\end{aligned}$$

□

Exercise 5.5.1 Let everything be as in Definition 5.2.1 a) and suppose that $u(0) = 0$. Prove then that

$$\|u\|^p_{L_{p,1}([0,T]\to E)} \leq C_T \int_0^T \|u'(s)\|^p_E ds.$$

Lemma 5.5.3 *Let $\beta > \beta(1,0)$. Then, $\{X^n\}_{n=1}^\infty$ are tight on $C([0,\infty) \to V_{2,-\beta})$.*

Proof. It is enough to prove the following for each fixed $T > 0$:

1) $(X_t^n)_{t \le T}$ $n = 1, 2, \dots$ are tight on $C([0,T] \to V_{2,-\beta})$.

To see this, we set:

$$\mathcal{S} = L_{2,1}([0,T] \to V_{2,-\beta(1,0)}) + L_{p,\alpha}([0,T] \to V_{2,0}), \text{ with } \alpha \in (0,1/2), p > 1/\alpha.$$

The idea is to take $\|\cdot\|_{\mathcal{S}}$ as the function F in Lemma 5.4.2. We have that

2) $$\sup_n E[\|X_0^n + J_\cdot^n\|_{L_{2,1}([0,T]\to V_{2,-\beta(1,0)})}] \overset{(5.20)}{\le} C_T < \infty$$

On the other hand,

3) $$\sup_n E[\|W_\cdot^n\|_{L_{p,\alpha}([0,T]\to V_{2,0})}] \overset{(5.16)}{\le} C_T < \infty.$$

We conclude from 2)–3) and the decomposition (5.19) that

$$\sup_n E[\|X_\cdot^n\|_{\mathcal{S}}] \le C_T < \infty$$

On the other hand, we see from Lemma 5.2.2 that

$$\mathcal{S} \hookrightarrow\hookrightarrow C([0,T] \to V_{2,-\beta}),$$

hence that the set:

$$\{X_\cdot\ ;\ \|X_\cdot^n\|_{\mathcal{S}} \le R\}$$

is relatively compact in $C([0,T] \to V_{2,-\beta})$. Thus, we have the tightness 1) by Lemma 5.4.2. □

Lemma 5.5.4 *Suppose that $\alpha \in [0,1)$. Then, $\{X^n\}_{n=1}^\infty$ are tight on $L_{2,\mathrm{loc}}([0,\infty) \to V_{2,\alpha})$.*

Proof. It is enough to prove the following for each fixed $T > 0$:

1) $(X_t^n)_{t \le T}$, $n = 1, 2, \dots$ are tight on $L_2([0,T] \to V_{2,\alpha})$.

To see this, we set:

$$\mathcal{I} = L_2([0,T] \to V_{2,1}) \cap L_{2,\gamma}([0,T] \to V_{2,-\beta(1,0)}), \text{ with } \gamma \in (0,1/2).$$

The idea is to take $\|\cdot\|_{\mathcal{I}}$ as the function F in Lemma 5.4.2. We have that

2) $$\sup_n E[\|X_\cdot^n\|^2_{L_2([0,T]\to V_{2,1})}] = \sup_n E[\int_0^T \|X_t^n\|^2_{2,1} dt] \overset{(5.10)}{\le} C_T < \infty$$

On the other hand,

$$\begin{aligned}
&\sup_n E[\|X_\cdot^n\|_{L_{2,\gamma}([0,T]\to V_{2,-\beta(1,0)})}] \\
&\quad \le \quad \sup_n E[\|X_0^n + J_\cdot^n\|_{L_{2,\gamma}([0,T]\to V_{2,-\beta(1,0)})}] + \sup_n E[\|W_\cdot^n\|_{L_{2,\gamma}([0,T]\to V_{2,0})}] \\
&\quad \overset{(5.16),(5.20)}{\le} \quad C_T < \infty.
\end{aligned}$$

We conclude from this and 2) that

$$\sup_n E[\|X^n_\cdot\|_{\mathcal{I}}] \le C_T < \infty.$$

On the other hand, we will see from Lemma 5.2.3 that

$$\mathcal{I} \hookrightarrow\hookrightarrow L_2([0,T] \to V_{2,\alpha}),$$

hence that the set :

$$\{X_\cdot\ ;\ \|X^n_\cdot\|_{\mathcal{I}} \le R\}$$

is relatively compact in $L_2([0,T] \to V_{2,\alpha})$. Thus, we have the tightness 1) by Lemma 5.4.2. □

Finally, Proposition 5.5.1 follows from Lemma 5.5.3–Lemma 5.5.4 and Lemma 5.4.4.

6 Proof of Theorem 3.2.1 and Theorem 3.2.2

6.1 Proof of Theorem 3.2.1

Let X and $\widetilde{X}^k$ be as in Proposition 5.5.1. We will verify that X takes values in the metric space (3.4) as well as properties (3.5)–(3.9) for X. (3.5) can easily be seen. In fact,

$$\begin{aligned} \widetilde{X}^k_0 &\to X_0 \quad \text{a.s. in } V_{2,-\beta}, \\ \widetilde{X}^k_0 \overset{\text{law}}{=} X^{n(k)}_0 = \mathcal{P}_{n(k)}\xi &\to \xi \quad \text{a.s. in } V_{2,0}. \end{aligned}$$

Thus the laws of X_0 and ξ are identical. To see (3.8)–(3.9), note that:

$$\|v_T\|_2^2, \quad \sup_{t\le T}\|v_t\|_2^2, \quad \int_0^T \|v_t\|_{2,1}^2 dt$$

are lower semi-continuous functions of $v_\cdot$ on the metric space (5.17). Thus, (3.8)–(3.9) follow from (5.10)–(5.11) and Proposition 5.5.1 via Fatou's lemma.

To show (3.6)–(3.7), we prepare the following:

Lemma 6.1.1 *Let $\varphi \in \mathcal{V}$ and $T > 0$. Then,*

$$\lim_{k\to\infty}\int_0^T \langle\, \varphi, (\widetilde{X}^k_t \cdot \nabla)\widetilde{X}^k_t \,\rangle dt = \int_0^T \langle\, \varphi, (X_t \cdot \nabla)X_t \,\rangle dt \ \ \textit{in probability}, \tag{6.1}$$

$$\lim_{k\to\infty}\int_0^T \langle\, \Delta\varphi, \widetilde{X}^k_t \,\rangle dt = \int_0^T \langle\, \Delta\varphi, X_t \,\rangle dt \ \ \textit{a.s.}, \tag{6.2}$$

$$\lim_{k\to\infty}\int_0^T \langle\, \varphi, \mathcal{P}_{n(k)} b(\widetilde{X}^k_t) \,\rangle dt = \int_0^T \langle\, \varphi, b(X_t) \,\rangle dt \ \ \textit{in probability}. \tag{6.3}$$

Proof: (6.1): Since,

$$\widetilde{X}^k_t \cdot \nabla\widetilde{X}^k_t - X_t \cdot \nabla X_t = (\widetilde{X}^k_t - X_t)\cdot\nabla\widetilde{X}^k_t + X_t \cdot \nabla(\widetilde{X}^k_t - X_t),$$

we have

$$\int_0^T |\langle\, \varphi, \widetilde{X}^k_t \cdot \nabla\widetilde{X}^k_t - X_t\cdot\nabla X_t \,\rangle| dt \le I_1 + I_2,$$

where

$$I_1 = \int_0^T |\langle \varphi, (\widetilde{X}_t^k - X_t) \cdot \nabla \widetilde{X}_t^k \rangle| dt, \text{ and } I_2 = \int_0^T |\langle \varphi, X_t \cdot \nabla(\widetilde{X}_t^k - X_t) \rangle| dt.$$

To bound I_1, we take

$$\alpha_1 = \alpha \in (0, 1 \wedge \tfrac{d}{2}), \quad \alpha_2 = 0, \quad \alpha_3 = \tfrac{d}{2} - \alpha \in (0, \tfrac{d}{2}).$$

in Lemma 2.2.1. Then, by (2.14), we have that

$$|\langle \varphi, (\widetilde{X}_t^k - X_t) \cdot \nabla \widetilde{X}_t^k \rangle| \leq C \|\widetilde{X}_t^k - X_t\|_{2,\alpha} \|\widetilde{X}_t^k\|_2 \|\varphi\|_{2,1+\alpha_3}$$

and hence that,

$$I_1 \leq C \|\varphi\|_{2,1+\alpha_3} \sup_{t \leq T} \|\widetilde{X}_t^k\|_2 \int_0^T \|\widetilde{X}_t^k - X_t\|_{2,\alpha} dt.$$

By (5.11) and Proposition 5.5.1,

$$\sup_{k \geq 1} E[\sup_{t \leq T} \|\widetilde{X}_t^k\|_2^2] < \infty \text{ and } \lim_{k \to \infty} \int_0^T \|\widetilde{X}_t^k - X_t\|_{2,\alpha}^2 dt = 0 \ \ P\text{-a.s.}$$

Then, it is easy to conclude from these that $\lim_{k \to \infty} I_1 = 0$ in probability (Exercise 6.1.1 below). To bound I_2, we take

$$\alpha_1 = 0, \quad \alpha_2 = \alpha \in (0, 1 \wedge \tfrac{d}{2}), \quad \alpha_3 = \tfrac{d}{2} - \alpha \in (0, \tfrac{d}{2})$$

in Lemma 2.2.1. On the other hand, we have by (2.14) that

$$|\langle \varphi, X_t \cdot \nabla(\widetilde{X}_t^k - X_t) \rangle| \leq C \|X_t\|_2 \|\widetilde{X}_t^k - X_t\|_{2,\alpha} \|\varphi\|_{2,1+\alpha_3}$$

and hence that,

$$I_2 \leq C \|\varphi\|_{2,1+\alpha_3} \sup_{t \leq T} \|X_t\|_2 \int_0^T \|\widetilde{X}_t^k - X_t\|_{2,\alpha} dt.$$

By (3.9) and Proposition 5.5.1,

$$E[\sup_{t \leq T} \|X_t\|_2^2] < \infty \text{ and } \lim_{k \to \infty} \int_0^T \|\widetilde{X}_t^k - X_t\|_{2,\alpha} dt = 0 \ \ P\text{-a.s.}$$

Then, it is easy to conclude from these that $\lim_{k \to \infty} I_2 = 0$ in probability (Exercise 6.1.1 below).
(6.2): This is an easy consequence of Proposition 5.5.1.
(6.3) follows from (6.1) and (6.2). Since $\varphi \in \mathcal{V}$ is fixed and k is tending to ∞, we do not have to care about $\mathcal{P}_{n(k)}$ here. $\Box$

Exercise 6.1.1 Let X_n, Y_n be r.v.'s such that $\{X_n\}_{n \geq 1}$ are tight and $Y_n \to 0$ in probability. Prove then that $X_n Y_n \to 0$ in probability.

We see (3.6)–(3.7) from the following:

Lemma 6.1.2 *Let:*

$$Y_t = Y_t(X) = X_t - X_0 - \int_0^t b(X_s)ds, \quad t \geq 0. \tag{6.4}$$

Then, $Y_\cdot$ is a $\mathrm{BM}(V_{2,0}, \Gamma)$. *Moreover, $Y_{t+\cdot} - Y_t$ and $\{\langle \varphi, X_s \rangle \ ; \ s \leq t, \varphi \in \mathcal{V}\}$ are independent for any $t \geq 0$.*

It is enough to prove that for each $\varphi \in \mathcal{V}$ and $0 \leq s < t$,

1) $$E\left[\exp\left(\mathbf{i}\langle \varphi, Y_t - Y_s \rangle\right) | \mathcal{G}_s\right] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\varphi \rangle\right), \text{ a.s.}$$

where $\mathcal{G}_s = \sigma(\langle \varphi, X_u \rangle \ ; \ u \leq s, \varphi \in \mathcal{V})$. We set

$$F(X) = f(\langle \varphi_1, X_{u_1} \rangle, ..., \langle \varphi_n, X_{u_n} \rangle),$$

where $f \in C_b(\mathbb{R}^n)$, $0 \leq u_1 < .. < u_n \leq s$ and $\varphi_1, ..., \varphi_n \in \mathcal{V}$ are chosen arbitrary in advance. Then, 1) can be verified by showing that

2) $$E\left[\exp\left(\mathbf{i}\langle \varphi, Y_t - Y_s \rangle\right) F(X)\right] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\varphi \rangle\right) E[F(X)].$$

Let:

$$Y_t^k = \widetilde{X}_t^k - \widetilde{X}_0^k - \int_0^t \mathcal{P}_{n(k)} b(\widetilde{X}_s^k)ds, \quad t \geq 0.$$

We then see from Theorem 5.1.1 that

3) $$E\left[\exp\left(\mathbf{i}\langle \varphi, Y_t^k - Y_s^k \rangle\right) F(\widetilde{X}^k)\right] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\mathcal{P}_{n(k)}\varphi \rangle\right) E[F(\widetilde{X}^k)],$$

Moreover, we have for any $\varphi \in \mathcal{V}$,

$$\lim_{k\to\infty} \langle \varphi, Y_t^k - Y_s^k \rangle \stackrel{(5.18),(6.3)}{=} \langle \varphi, Y_t - Y_s \rangle \quad \text{in probability,}$$

and hence

$$\lim_{k\to\infty} \text{LHS of 3)} = \text{ LHS of 2)}.$$

On the other hand,

$$\lim_{k\to\infty} \text{ RHS of 3)} \stackrel{(5.18)}{=} \text{ RHS of 2)}.$$

These prove 2). □

Finally, we prove that X takes values in the metric space (3.4). It follows from (3.9) that

$$X \in L_{2,\mathrm{loc}}([0,\infty) \to V_{2,1}) \cap L_{\infty,\mathrm{loc}}([0,\infty) \to V_{2,0}).$$

Thus, it remains to show that $X \in C([0,\infty) \to V_{2,-\beta(1,1)})$. We see from Lemma 2.2.3 that:

$$\int_0^\cdot b(X_s)ds \in C([0,\infty) \to V_{2,-\beta(1,1)}) \text{ if } X \in L_2([0,\infty) \to V_{2,1}).$$

On the other hand, $Y \in C([0,\infty) \to V_{2,0})$. These show that $X \in C([0,\infty) \to V_{2,-\beta(1,1)})$. □

6.2 Proof of Theorem 3.2.2

Here, we can follow the argument of [Te79, p. 294, Theorem 3.2] almost verbatim. We will present it for the convenience of the readers.

We need technical lemmas:

Lemma 6.2.1 *[Te79, pp. 60–61, Lemma 1.2] Let H and and V be a Hilbert spaces such that:*

$$V \hookrightarrow H \hookrightarrow V^*.$$

Suppose that $f \in L_2([0,T] \to V)$ has derivative f' in $L_2([0,T] \to V^*)$. Then,

$$\frac{d}{dt}|f|_H^2 = 2_V\langle\, f, f'\,\rangle_{V^*}, \tag{6.5}$$

in the distributional sense on $(0,T)$.

Lemma 6.2.2 *For any $T > 0$, there exists $C_T \in (0,\infty)$ such that:*

$$E\left[\int_0^T \|b(X_t)\|_{2,-\beta(1,0)}\right] \le C_T < \infty. \tag{6.6}$$

Proof: Using (3.9), the lemma can be shown in the same way as Lemma 5.5.2. □

Let X and $\widetilde{X}$ be as in the assumptions of Theorem 3.2.2 and

$$Z_t = X_t - \widetilde{X}_t = \int_0^t \left(b(X_s) - b(\widetilde{X}_s)\right) ds.$$

Then,

1) $\quad Z_\cdot \in L_{2,\mathrm{loc}}([0,\infty) \to V_{2,1})$

and by Lemma 6.2.2,

2) $\quad \partial_t Z_\cdot = b(X_\cdot) - b(\widetilde{X}_\cdot) \in L_{2,\mathrm{loc}}([0,\infty) \to V_{2,-\beta(1,0)})$

Since $\beta(1,0) = 1$, we see from 2) and Lemma 6.2.1 (applied to $f = Z_\cdot$ and $V = V_{2,1}$) that

3) $\quad \frac{1}{2}\frac{d}{dt}\|Z_t\|_2^2 \stackrel{(6.5)}{=} \langle\, Z_t, b(X_t) - b(\widetilde{X}_t)\,\rangle = -I_t - J_t$

in the distributional sense, where

$$\begin{aligned} I_t &= \langle\, Z_t, (X_t\cdot\nabla)X_t - (\widetilde{X}_t\cdot\nabla)\widetilde{X}_t\,\rangle, \\ J_t &= \nu\langle\,\nabla Z_t, \nabla X_t - \nabla\widetilde{X}_t\,\rangle = \nu\|\nabla Z_t\|_2^2. \end{aligned}$$

On the other hand, since $\widetilde{X}_t = X_t - Z_t$, we see that

$$\langle\, Z_t, (\widetilde{X}_t\cdot\nabla)\widetilde{X}_t\,\rangle \stackrel{\text{Lemma 2.1.2}}{=} \langle\, Z_t, (\widetilde{X}_t\cdot\nabla)X_t\,\rangle = \langle\, Z_t, ((X_t - Z_t)\cdot\nabla)X_t\,\rangle,$$

and hence that

$$I_t = \langle\, Z_t, (Z_t\cdot\nabla)X_t\,\rangle.$$

We now apply Lemma 2.2.2 with $(\alpha_1,\alpha_2,\alpha_3) = (1,0,0)$. Note that these α_i satisfy the assumption of Lemma 2.2.2 only when $d = 2$.

4) $$|I_t| \le C_3\|Z_t\|_{2,1}\|Z_t\|_2\|X_t\|_{2,1} \le \nu\|Z_t\|_{2,1}^2 + C_4\|X_t\|_{2,1}^2\|Z_t\|_2^2.$$

We see from 3)–4) that

$$\tfrac{1}{2}\tfrac{d}{dt}\|Z_t\|_2^2 \le C_4\|X_t\|_{2,1}^2\|Z_t\|_2^2.$$

This implies, via Gronwall's lemma (We need an appropriate generalization, since the derivative above is in the distributional sense.) that

$$\|Z_t\|_2^2 \le \|Z_0\|_2^2 \exp\left(C_4\int_0^t \|X_s\|_{2,1}^2 ds\right).$$

This proves that $\|Z_t\|_2 \equiv 0$. □

7 Appendix

Lemma 7.0.3 *Suppose that a CONS $\{\varphi_n\}_{n\ge1}$ of H and numbers $\gamma_n \ge 0$ satisfy (5.14).*

a) *Let $\{B^k_\cdot\}_{k\in\mathbb{N}}$ be independent standard* BM[1]*'s. Then, the process*

$$W_t^n = \sum_{k=0}^n \sqrt{\gamma_k} B_t^k \varphi_k, \quad t \ge 0, \tag{7.1}$$

converges to a BM(H,Γ) $W_\cdot$ in the sense that:

$$\lim_{n\to\infty} E\left[\sup_{t\le T}\|W_t^n - W_t\|^2\right] = 0 \quad \textit{for any } T > 0. \tag{7.2}$$

b) *For any BM(H,Γ) $W_\cdot$, there are independent standard* BM[1]*'s such that (7.2) holds with the process defined by (5.15).*

Proof: a): Let us show that

1) $(W^n_\cdot)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to seminorms:

$$|||W|||_t = E\left[\sup_{s\le t}\|W_s\|^2\right]^{1/2}, \quad t \in (0,\infty).$$

In fact, for $m < n$,

$$\|W_s^n - W_s^m\|^2 = \sum_{m<k\le n} \gamma_k |B_s^k|^2.$$

By this and Doob's L^2-maximal inequality,

$$E\left[\sup_{s\le t}\|W_s^n - W_s^m\|^2\right] \le \sum_{m<k\le n}\gamma_k E\left[\sup_{s\le t}|B_s^k|^2\right] \stackrel{(4.16)}{\le} 4t\sum_{m<k\le n}\gamma_k \stackrel{m\to\infty}{\longrightarrow} 0.$$

By 1), there exists a random variable W with values in $C([0,\infty)\to H)$ such that (7.2) holds. It is easy to see from this that for $0 \le s < t$:

$$\lim_{n\to\infty} \exp\left(\mathbf{i}\langle \varphi, W_t^n - W_s^n\rangle\right) = \exp\left(\mathbf{i}\langle \varphi, W_t - W_s\rangle\right) \text{ in } L^1(P),$$

and hence

2) $$\lim_{n\to\infty} E\left[\exp\left(\mathbf{i}\langle \varphi, W_t^n - W_s^n \rangle\right) | \mathcal{G}_s^W\right] = E\left[\exp\left(\mathbf{i}\langle \varphi, W_t - W_s \rangle\right) | \mathcal{G}_s^W\right] \quad \text{in } L^1(P).$$

On the other hand,

$$\begin{aligned} E\left[\exp\left(\mathbf{i}\langle \varphi, W_t^n - W_s^n \rangle\right) | \mathcal{G}_s^W\right] &= E\left[\exp\left(\mathbf{i}\langle \varphi, W_t^n - W_s^n \rangle\right)\right] \\ &= \prod_{k=0}^n E\left[\exp\left(\mathbf{i}\sqrt{\gamma_k}\langle \varphi, \varphi_k \rangle (B_t^k - B_s^k)\right)\right] \\ &= \prod_{k=0}^n \exp\left(-\frac{t-s}{2}\gamma_k \langle \varphi, \varphi_k \rangle^2\right) \stackrel{n\to\infty}{\longrightarrow} \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\varphi \rangle\right). \end{aligned}$$

By this and 2), we have (3.3).
b): Processes:

$$B_\cdot^k \stackrel{\text{def}}{=} \langle W_\cdot, \varphi_k \rangle / \sqrt{\gamma_k}, \quad k \in I \stackrel{\text{def}}{=} \{k \in \mathbb{N}\ ;\ \gamma_k > 0\}$$

are independent BM^1's. Let $\{B_\cdot^k\}_{k\in N\setminus I}$ be independent BM^1's which are independent of $\{B_\cdot^k\}_{k\in I}$. Then, $\langle W_\cdot, \varphi_k \rangle = \sqrt{\gamma_k} B_\cdot^k$ for all $k \in \mathbb{N}$, and hence (5.15) holds. □

Acknowledgements: This article was originally written for a course at Kyoto University. I am grateful for Professor Hayato Nawa for an opprtunity to present a talk on the subject of this article.

References

[IW89] Ikeda, N. and Watanabe, S. : Stochastic Differential Equations and Diffusion Processes (2nd ed.), North-Holland, Amsterdam / Kodansha,Tokyo (1989).

[Fl08] Flandoli, Franco : An introduction to 3D stochastic fluid dynamics. SPDE in hydrodynamic: recent progress and prospects, 51–150, Lecture Notes in Math., 1942, Springer, Berlin, 2008.

[FG95] Flandoli, Franco ; Gatarek, Dariusz: Martingale and stationary solutions for stochastic Navier-Stokes equations. Probab. Theory Related Fields 102 (1995), no. 3, 367–391.

[Ho50] Hopf, Eberhard : Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichngen. Math. Nach. **4** (1950), 213–231.

[KS91] Karatzas, I. and Shreve, S. E.: Brownian Motion and Stochastic Calculus, Second Edition. Springer Verlag (1991).

[Le33] Leray, Jean : Étude de diverse équation intégrales non linéares et de quelques problémes que pose l'hydrodynamique. J. Math. Pure. Appl. **12**, (1933), 1–82.

[Le34a] Leray, Jean : Sur le mouvement d'un liquide visqueux emplissant l'espace. (French) Acta Math. 63 (1934), no. 1, 193–248.

[Le34b] Leray, Jean : Essai sur les mouvements d'un liquide visqueux que limitent des paroi. J. de Math. **XIII** (1934), 331–418.

[RS72] Reed, M. and Simon, B. "Method of Modern Mathematical Physics I: Functional Analysis" Academic Press 1980.

[Ta96] Taylor, M. E. : Partial Differential Equations III, Springer-Verlag New York Berlin Heiderberg (1996).

[Te79] Temam, Roger: Navier-Stokes Equations. North-Holland Publishing Company (1979).